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RENORMALIZATION OF THE STANDARD MODEL

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Contents

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1 Introduction

The e^+e^- colliders LEP and SLC have opened a new era of precision experiments. The determination of the Z resonance parameters with high accuracy by the four LEP experiments [1], together with the measurement of the W mass at $p\bar{p}$ colliders [2] and with the data from neutrino scattering experiments [3] allow precision tests of the electroweak theory never reached before [4]. The present theory of the electroweak interaction, known as the "Standard Model", is the Glashow–Salam-Weinberg model [5] of leptons, extended via the GIM mechanism [6] to the hadronic sector thus incorporating the idea of Cabibbo mixing [7], and made anomaly free through the introduction of the concept of color [8]. As such it is the most comprehensive description of the electroweak phenomena being theoretically consistent and in extraordinary agreement with the experimental data [4].

The possibility of performing precision tests of the electroweak Standard Model is essentially based on its formulation in terms of a quantum field theory with spontaneous symmetry breaking which is renormalizable [9] and thus allows perturbative calculations for measureable quantities order by order in terms of a few input parameters. The input parameters themselves cannnot be predicted but have to be taken from appropriate experiments. Comparison of the theoretical predictions with the results from precision experiments has confirmed the validity of the Standard Model as a fully fledged quantum field theory, in complete analogy to what has been done for QED. Signals giving need for some significant modifications have not yet been observed.

The still inherent uncertainties in the predictions are related to the as yet unknown mass parameters of the top quark and the Higgs boson. The experimental lower bound for the top mass has improved from 91 GeV [10] to 103 GeV (D0) and 108 GeV (CDF) at 95% C.L. [11], and the existence of a standard Higgs boson can be excluded in the mass range below 60 GeV [4]. According to the principles of quantum field theory, the virtual presence of all physical states in the spectrum shows up in higher order calculations. As a consequence, the experimentally unobserved particles top and Higgs affect the theoretical predictions for the various physical quantities in a calculable way. The same ideas apply, in principle, to all kinds of objects connected with structures beyond the "minimal" standard model (like more Higgs fields, supersymmetric partners, new vector bosons,...) which are too heavy for being observed directly in the experimental search. The higher order contributions hence simultaneously represent an important window to "new physics", in particular to those situations where the decoupling theorem [12] does not hold automatically. One may also consider modifications of the tree level Lagrangian with the interesting possibility of small zeroth order deviations which mimic or cancel the effect of radiative corrections. Since deviations from the Standard Model predictions have not been observed, possible theoretically motivated extensions of the minimal model are subject to significant constraints.

The higher order terms in the perturbative expansion of physical quantities, or radiative corrections, arise from the quantum structure of the underlying theory. Higher order effects related to the presence of the Higgs particle, the top quark, as well as to the self interaction of the gauge fields represent the "genuine" radiative corrections of the electroweak theory. Their typical size in observable quantities is expected to be $\delta_W \leq O(10^{-2})$. Their observation requires therefore that the theoretical uncertainties are not bigger than 0.1%. This requires to go beyond the one-loop approximation and to include at least the leading contributions from the next order. Also the QED corrections, common to any theory containing the electromagnetic U(1) subgroup, have to be treated carefully and have to be clearly disentangled from the genuine weak corrections in the theoretical predictions and in the analysis of the experiments.

The electroweak Standard Model contains, besides fermion masses, quark mixing angles, and the mass of the Higgs scalar, three free parameters in the gauge sector. In order to make predictions for processes mediated by the exchange of gauge bosons, three independent experimental input data are required for fixing the SU(2) and U(1) gauge coupling constants g_2, g_1, and the vacuum expectation value v of the Higgs field. It is, however, more practical to deal with parameters such that each of them has a direct relation to a specific experiment and is a well measured quantity. The most accurate set of data points consists of the electromagnetic fine structure constant [13] $\alpha = 137.0359895(6)^{-1}$, the Fermi constant [13] $G_\mu = 1.16639(2) \cdot 10^{-5}$ GeV^{-2}, and the mass of the Z boson $M_Z = 91.187 \pm 0.007$ GeV [1]. The masses of the Higgs boson and the top quark enter the higher order calculations as additional free parameters unless a direct experimental determination of their values is available. Since also the strong interaction is present in the higher order contributions to electroweak quantities, the strong coupling constant α_s is a further independent input parameter.

The calculation of radiative corrections is a lenghty and involved task. Before predictions can be made, a careful discussion of regularization and renormalization is required, together with the extensive use of techniques for the evaluation of loop diagrams. It is the purpose of this presentation to give a comprehensive description of the basic entries and concepts for the calculation of electroweak observables beyond the lowest order and to provide the reqired expressions for practical calculations, with special emphasis on the vector boson masses and e^+e^- processes. To begin with, we briefly recall the basic Lagrangian of the electroweak Standard Model and its parametrizations, as the starting point for perturbative calculations (section 2).

Section 3 is concerned with the concept of renormalization and with a detailed discussion of the renormalization in the on-shell scheme which treats the particle masses together with α as the basic parameters in the perturbation expansion. The results can be considered as the building blocks to be used for the computation of amplitudes for electroweak processes at the 1-loop level. Technical details for the evaluation of electroweak 1-loop diagrams are collected in section 4.

In section 5 we provide the explicit 1-loop results for the Standard Model self energies of fermions and gauge bosons. This completes the 1-loop renormalization and allows us to discuss the impact of parameter renormalization on the correlation between the various electroweak quantities. The detailed interdependence of the electroweak parameters in terms of the Fermi constant is presented in section 6. It enables us to predict the M_W-M_Z correlation and to perform comparisons with existing data thereby setting bounds to the range of the unknown top and Higgs masses. The necessity of including higher than 1-loop order terms is discussed, together with the description of how the 1-loop results have to be modified in order to incorporate all the next order terms available from existing calculations. This is accompanied by a discussion of the remaining uncertainties in theoretical predictions.

Section 7 contains a review of other renormalization schemes. In particular we describe the $\overline{MS}$ renormalization scheme in some more detail and give the relation between the $\overline{MS}$ and the on-shell parameters, together with numerical results. The last section 8 is devoted to renormalizable extensions of the minimal model.

Applications to e^+e^- processes are considered separately in the subsequent extra chapter.

2 The tree level Lagrangian

2.1 *The classical Lagrangian*

The phenomenological basis for the formulation of the Standard Model is given by the following empirical facts:

- The SU(2)×U(1) family structure of the fermions:
  The fermions appear as families with left-handed doublets and right-handed singlets:

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \quad e_R, \quad \mu_R, \quad \tau_R$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b \end{pmatrix}_L, \quad u_R, \quad d_R, \quad c_R, \cdots$$

  They can be characterized by the quantum numbers of the weak isospin I, I_3, and the weak hypercharge Y.

- The Gell-Mann-Nishijima relation:
  Between the quantum numbers classifying the fermions with respect to the group SU(2)×U(1) and their electric charges Q the relation

$$Q = I_3 + \frac{Y}{2} \tag{1}$$

  is valid.

- The existence of vector bosons:
  There are 4 vector bosons as carriers of the electroweak force

$$\gamma, \quad W^+, \quad W^-, \quad Z$$

  where the photon is massless and the $W^\pm$, Z have masses $M_W \neq 0$, $M_Z \neq 0$.

This empirical structure can be embedded in a gauge invariant field theory of the unified electromagnetic and weak interactions by interpreting SU(2)×U(1) as the group of gauge transformations under which the Lagrangian is invariant. This full symmetry has to be broken by the Higgs mechanism down to the electromagnetic gauge symmetry; otherwise the $W^\pm$, Z bosons would also be massless. The minimal formulation, the Standard Model, requires a single scalar field (Higgs field) which is a doublet under SU(2).

According to the general principles of constructing a gauge invariant field theory with spontaneous symmetry breaking, the gauge, Higgs, and fermion parts of the electroweak Lagrangian

$$\mathcal{L}_{cl} = \mathcal{L}_G + \mathcal{L}_H + \mathcal{L}_F \tag{2}$$

are specified in the following way:

Gauge fields
SU(2)×U(1) is a non-Abelian group which is generated by the isospin operators I_1, I_2, I_3 and the hypercharge Y (the elements of the corresponding Lie algebra). Each of these generalized charges is associated with a vector field: a triplet of vector fields $W_\mu^{1,2,3}$ with $I_{1,2,3}$ and a singlet field B_μ with Y. The isotriplet W_μ^a, $a = 1, 2, 3$, and the isosinglet B_μ lead to the field strength tensors

$$\begin{aligned} W_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g_2\, \epsilon_{abc}\, W_\mu^b W_\nu^c, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu. \end{aligned} \tag{3}$$

g_2 denotes the non-Abelian SU(2) gauge coupling constant and g_1 the Abelian U(1) coupling. From the field tensors (3) the pure gauge field Lagrangian

$$\mathcal{L}_G = -\frac{1}{4}\, W_{\mu\nu}^a W^{\mu\nu,a} - \frac{1}{4}\, B_{\mu\nu} B^{\mu\nu} \tag{4}$$

is formed according to the rules for the non-Abelian case.

Fermion fields and fermion-gauge interaction
The left-handed fermion fields of each lepton and quark family (colour index is suppressed)

$$\psi_j^L = \begin{pmatrix} \psi_{j+}^L \\ \psi_{j-}^L \end{pmatrix}$$

with family index j are grouped into SU(2) doublets with component index $\sigma = \pm$, and the right-handed fields into singlets

$$\psi_j^R = \psi_{j\sigma}^R .$$

Each left- and right-handed multiplet is an eigenstate of the weak hypercharge Y such that the relation (1) is fulfilled. The covariant derivative

$$D_\mu = \partial_\mu - i\, g_2\, I_a W_\mu^a + i\, g_1 \frac{Y}{2} B_\mu \tag{5}$$

induces the fermion-gauge field interaction via the minimal substitution rule:

$$\mathcal{L}_F = \sum_j \bar{\psi}_j^L i\gamma^\mu D_\mu \psi_j^L + \sum_{j,\sigma} \bar{\psi}_{j\sigma}^R i\gamma^\mu D_\mu \psi_{j\sigma}^R \tag{6}$$

Higgs field, Higgs - gauge field and Yukawa interaction
For spontaneous breaking of the SU(2)×U(1) symmetry leaving the electromagnetic gauge subgroup $U(1)_{em}$ unbroken, a single complex scalar doublet field with hypercharge $Y = 1$

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix} \tag{7}$$

is coupled to the gauge fields

$$\mathcal{L}_H = (D_\mu \Phi)^+ (D^\mu \Phi) - V(\Phi) \tag{8}$$

with the covariant derivative

$$D_\mu = \partial_\mu - i\, g_2\, I_a W_\mu^a + i \frac{g_1}{2} B_\mu .$$

The Higgs field self-interaction

$$V(\Phi) = -\mu^2\, \Phi^+ \Phi + \frac{\lambda}{4} (\Phi^+ \Phi)^2 \tag{9}$$

is constructed in such a way that it has a non-vanishing vacuum expectation value v, related to the coefficients of the potential V by

$$v = \frac{2\mu}{\sqrt{\lambda}} . \tag{10}$$

The field (7) can be written in the following way:

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ (v + H(x) + i\chi(x))/\sqrt{2} \end{pmatrix} \tag{11}$$

where the components ϕ^+, H, χ now have vacuum expectation values zero. Exploiting the invariance of the Lagrangian one notices that the components ϕ^+, χ can be gauged away which means that they are unphysical (Higgs ghosts or would-be Goldstone bosons). In this particular gauge, the unitary gauge, the Higgs field has the simple form

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} .$$

The real part of ϕ^0, $H(x)$, describes physical neutral scalar particles with mass

$$M_H = \mu\sqrt{2}. \tag{12}$$

The Higgs field components have triple and quartic self couplings following from V, and couplings to the gauge fields via the kinetic term of Eq. (8).

In addition, Yukawa couplings to fermions are introduced in order to make the charged fermions massive. The Yukawa term is conveniently expressed in the doublet field components (7). We write it down for one family of leptons and quarks:

$$\begin{aligned} \mathcal{L}_{Yukawa} &= -g_l\,(\overline{\nu}_L\,\phi^+\,l_R + \overline{l}_R\,\phi^-\,\nu_L + \overline{l}_L\,\phi^0\,l_R + \overline{l}_R\,\phi^{0*}\,l_L) \\ &= -g_d\,(\overline{u}_L\,\phi^+\,d_R + \overline{d}_R\,\phi^-\,u_L + \overline{d}_L\,\phi^0\,d_R + \overline{d}_R\,\phi^{0*}\,d_L) \\ &\quad - g_u\,(\overline{u}_R\,\phi^+\,d_L + \overline{d}_L\,\phi^-\,u_R + \overline{u}_R\,\phi^0\,u_L + \overline{u}_L\,\phi^{0*}\,u_R)\,. \end{aligned} \tag{13}$$

ϕ^- denotes the adjoint of ϕ^+.
By $v \neq 0$ fermion mass terms are induced. The Yukawa coupling constants $g_{l,d,u}$ are related to the masses of the charged fermions by Eq. (23). In the unitary gauge the Yukawa Lagrangian is particularly simple:

$$\mathcal{L}_{Yukawa} = -\sum_f m_f\,\bar{\psi}_f\psi_f - \sum_f \frac{m_f}{v}\,\bar{\psi}_f\psi_f\,H\,. \tag{14}$$

As a remnant of this mechanism for generating fermion masses in a gauge invariant way, Yukawa interactions between the massive fermions and the physical Higgs field occur with coupling constants proportional to the fermion masses.

Physical fields and parameters

The gauge invariant Higgs-gauge field interaction in the kinetic part of Eq. (8) gives rise to mass terms for the vector bosons in the non-diagonal form

$$\frac{1}{2}\left(\frac{g_2}{2}v\right)^2 (W_1^2 + W_2^2) + \frac{v^2}{4}\left(W_\mu^3, B_\mu\right) \begin{pmatrix} g_2^2 & g_1 g_2 \\ g_1 g_2 & g_1^2 \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} . \tag{15}$$

The physical content becomes transparent by performing a transformation from the fields W_μ^a, B_μ (in terms of which the symmetry is manifest) to the "physical" fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm iW_\mu^2) \tag{16}$$

and

$$\begin{aligned} Z_\mu &= +\cos\theta_W\, W_\mu^3 + \sin\theta_W\, B_\mu \\ A_\mu &= -\sin\theta_W\, W_\mu^3 + \cos\theta_W\, B_\mu \end{aligned} \tag{17}$$

In these fields the mass term (15) is diagonal and has the form

$$M_W^2\, W_\mu^+ W^{-\mu} + \frac{1}{2}(A_\mu, Z_\mu)\begin{pmatrix} 0 & 0 \\ 0 & M_Z^2 \end{pmatrix}\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} \tag{18}$$

with

$$\begin{aligned} M_W &= \frac{1}{2} g_2 v \\ M_Z &= \frac{1}{2}\sqrt{g_1^2 + g_2^2}\, v \end{aligned} \tag{19}$$

The mixing angle in the rotation (17) is given by

$$\cos\theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} = \frac{M_W}{M_Z}\,. \tag{20}$$

Identifying A_μ with the photon field which couples via the electric charge $e = \sqrt{4\pi\alpha}$ to the electron, e can be expressed in terms of the gauge couplings in the following way

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} \tag{21}$$

or

$$g_2 = \frac{e}{\sin\theta_W}, \quad g_1 = \frac{e}{\cos\theta_W}. \tag{22}$$

Finally, from the Yukawa coupling terms in Eq. (13) the fermion masses are obtained:

$$m_f = g_f \frac{v}{\sqrt{2}} = \sqrt{2}\,\frac{g_f}{g_2}\, M_W\,. \tag{23}$$

The relations above allow one to replace the original set of parameters

$$g_2,\ g_1,\ \lambda,\ \mu^2,\ g_f \tag{24}$$

by the equivalent set of more physical parameters

$$e,\ M_W,\ M_Z,\ M_H,\ m_f \tag{25}$$

where each of them can (in principle) directly be measured in a suitable experiment.

An additional very precisely measured parameter is the Fermi constant G_μ which is the effective 4-fermion coupling constant in the the Fermi model, measured by the muon lifetime:

$$G_\mu = 1.16639(2) \cdot 10^{-5}\,\mathrm{GeV}^{-2}$$

Consistency of the Standard Model at $q^2 \ll M_W^2$ with the Fermi model requires the identification (see section 6)

$$\frac{G_\mu}{\sqrt{2}} = \frac{e^2}{8\sin^2\theta_W M_W^2}, \tag{26}$$

which allows us to relate the vector boson masses to the parameters α, G_μ, and $\sin^2\theta_W$ as follows:

$$\begin{aligned} M_W^2 &= \frac{\pi\alpha}{\sqrt{2}G_\mu}\cdot\frac{1}{\sin^2\theta_W} \\ M_Z^2 &= \frac{\pi\alpha}{\sqrt{2}G_\mu}\cdot\frac{1}{\sin^2\theta_W\cos^2\theta_W} \end{aligned} \tag{27}$$

and thus to establish also the $M_W - M_Z$ interdependence:

$$M_W^2\left(1-\frac{M_W^2}{M_Z^2}\right) = \frac{\pi\alpha}{\sqrt{2}G_\mu}. \tag{28}$$

2.2 *Gauge fixing and ghost fields*

Since the S matrix element for any physical process is a gauge invariant quantity it is possible to work in the unitary gauge with no unphysical particles in internal lines. For a systematic treatment of the quantization of $\mathcal{L}_{cl}$ and for higher order calculations, however, one better refers to a renormalizable gauge. This can be done by adding to $\mathcal{L}_{cl}$ a gauge fixing Lagrangian, for example

$$\mathcal{L}_{fix} = -\frac{1}{2}\left(F_\gamma^2 + F_Z^2 + 2F_+F_-\right) \tag{29}$$

with linear gauge fixings of the 't Hooft type:

$$\begin{aligned} F_\pm &= \frac{1}{\sqrt{\xi^W}}\left(\partial^\mu W_\mu^\pm \mp iM_W\xi^W\phi^\pm\right) \\ F_Z &= \frac{1}{\sqrt{\xi^Z}}\left(\partial^\mu Z_\mu - M_Z\xi^Z\chi\right) \\ F_\gamma &= \frac{1}{\sqrt{\xi^\gamma}}\partial^\mu A_\mu \end{aligned} \tag{30}$$

with arbitrary parameters $\xi^{W,Z,\gamma}$. In this class of 't Hooft gauges, the vector boson propagators have the form

$$\frac{i}{k^2 - M_V^2}\left(-g^{\mu\nu} + \frac{(1-\xi^V)k^\mu k^\nu}{k^2 - \xi^V M_V^2}\right)$$

$$= \frac{i}{k^2 - M_V^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right) + \frac{i\xi^V}{k^2 - \xi^V M_V^2}\,\frac{k^\mu k^\nu}{k^2}, \tag{31}$$

the propagators for the unphysical Higgs fields are given by

$$\frac{i}{k^2 - \xi^W M_W^2} \quad \text{for} \quad \phi^\pm \tag{32}$$

$$\frac{i}{k^2 - \xi^Z M_Z^2} \quad \text{for} \quad \chi^0, \tag{33}$$

and Higgs-vector boson transitions do not occur.

For completion of the renormalizable Lagrangian the Faddeev-Popov ghost term $\mathcal{L}_{gh}$ has to be added [14] in order to balance the undesired effects in the unphysical components introduced by $\mathcal{L}_{fix}$:

$$\mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{fix} + \mathcal{L}_{gh} \tag{34}$$

where

$$\mathcal{L}_{gh} = \bar{u}^\alpha(x)\,\frac{\delta F^\alpha}{\delta\theta^\beta(x)}\,u^\beta(x) \tag{35}$$

with ghost fields u^γ, u^Z, $u^\pm$, and $\frac{\delta F^\alpha}{\delta\theta^\beta}$ being the change of the gauge fixing operators (35) under infinitesimal gauge transformations characterized by $\theta^\alpha(x) = \{\theta^a(x),\, \theta^Y(x)\}$.

In the 't Hooft-Feynman gauge ($\xi = 1$) the vector boson propagators (31) become particularly simple: the transverse and longitudinal components, as well as the propagators for the unphysical Higgs fields $\phi^\pm$, χ and the ghost fields $u^\pm$, u^Z have poles which coincide with the masses of the corresponding physical particles $W^\pm$ and Z.

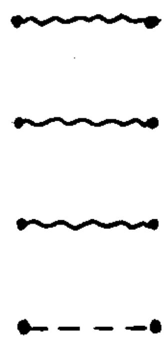
2.3 *Feynman rules*

Expressed in terms of the physical parameters we can write down the Lagrangian

$$\mathcal{L}(A_\mu, W_\mu^\pm, Z_\mu, H, \phi^\pm, \chi, u^\pm, u^Z, u^\gamma;\; M_W, M_Z, e, \ldots)$$

in a way which allows us to read off the propagators and the vertices most directly. We specify them in the $R_{\xi=1}$ gauge where the vector boson propagators have the simple algebraic form $\sim g_{\mu\nu}$.

$$\mathcal{L}_G + \mathcal{L}_H =$$



$$\frac{1}{2} A_\mu \Box A^\mu \qquad \frac{-i\, g_{\mu\nu}}{q^2}$$

$$+ W_\mu^- (\Box + M_W^2)\, W^{+\mu} \qquad \frac{-i\, g_{\mu\nu}}{q^2 - M_W^2}$$

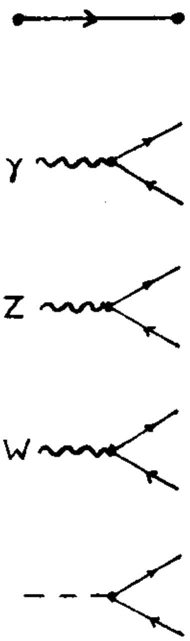
$$+ \frac{1}{2} Z_\mu (\Box + M_Z^2)\, Z^\mu \qquad \frac{-i\, g_{\mu\nu}}{q^2 - M_Z^2}$$

$$+ \frac{1}{2} H\, (\Box + M_H^2)\, H \qquad \frac{i}{q^2 - M_H^2}$$

+ interaction terms
$VV,\ VH,\ HH$

+ (unphysical degrees of freedom)

$$\mathcal{L}_F + \mathcal{L}_{Yukawa} =$$



$$\sum_f \bar{f}\,(i\partial - m_f)\, f \qquad \frac{i}{q - m_f}$$

$$+ J^\mu_{em} A_\mu \qquad -i\, e\, Q_f \gamma_\mu$$

$$+ J^\mu_{NC} Z_\mu \qquad i\, \frac{e}{2 \sin\theta_W \cos\theta_W}\, \gamma_\mu (v_f - a_f \gamma_5)$$

$$+ J^\mu_{CC} W_\mu \qquad i\, \frac{e}{2\sqrt{2} \sin\theta_W}\, \gamma_\mu (1 - \gamma_5)\, V_{jk}$$

$$- \frac{g_f}{\sqrt{2}}\, \bar{f} f\, H \qquad -i\, \frac{g_f}{\sqrt{2}} = i\, \frac{e}{2 \sin\theta_W}\, \frac{m_f}{M_W}$$

+ (unphysical degrees of freedom) (36)

These Feynman rules provide the ingredients to calculate the lowest order amplitudes for fermionic processes. For the complete list of all interaction vertices we refer to the literature [15].

In order to describe scattering processes between light fermions in lowest order we can, in most cases, neglect the exchange of Higgs bosons because of their small Yukawa couplings to the known fermions. The standard processes accessible by the experimental facilities are basically 4-fermion processes. These are mediated by the gauge bosons and, sufficient in lowest order, defined by the vertices for the fermions interacting with the vector bosons. They are given in the Lagrangian above for the electromagnetic, neutral and charged current interactions. The neutral current coupling constants in (36) read

$$\begin{aligned} v_f &= I_3^f - 2Q_f \sin^2\theta_W \\ a_f &= I_3^f \,. \end{aligned} \tag{37}$$

Q_f and I_3^f denote the charge and the third isospin component of f_L.

The quantities V_{jk} in the charged current vertex are the elements of the unitary 3×3 matrix

$$U_{KM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{38}$$

which describes family mixing in the quark sector [7]. Its origin is the diagonalization of the quark mass matrices from the Yukawa coupling which appears since quarks of the same charge have different masses. For massless neutrinos no mixing in the leptonic sector is present. Due to the unitarity of U_{KM} the mixing is absent in the neutral current.

For a proper treatment of the charged current vertex at the one-loop level, the matrix U_{KM} has to be renormalized as well. As it was shown in [16], where the renormalization procedure was extended to U_{KM}, the resulting effects are completely negligible for the known light fermions. We therefore skip the renormalization of U_{KM} in our discussion of radiative corrections.

3 Renormalization

3.1 *General remarks*

The tree level Lagrangian (2) of the minimal SU(2)×U(1) model involves a certain number of free parameters which are not fixed by the theory. The definition of these parameters and their relation to measurable quantities is the content of a renormalization scheme. The parameters (or appropriate combinations) can be determined from specific experiments with help of the theoretical results for cross sections and lifetimes. After this procedure of defining the physical input, other observables can be predicted allowing verification or falsification of the theory by comparison with the corresponding experimental results.

In higher order perturbation theory the relations between the formal parameters and measureable quantities are different from the tree level relations in general. Moreover, the procedure is obscured by the appearence of divergences from the loop integrations. For a mathematically consistent treatment one has to regularize the theory, e.g. by dimensional regularization (performing the calculations in D dimensions). But then the relations between physical quantities and the parameters become cutoff dependent. Hence, the parameters of the basic Lagrangian, the "bare" parameters, have no physical meaning. On the other hand, relations between measureable physical quantities, where the parameters drop out, are finite and independent of the cutoff. It is therefore in principle possible to perform tests of the theory in terms of such relations by eliminating the bare parameters [17, 73].

Alternatively, one may replace the bare parameters by renormalized ones by multiplicative renormalization for each bare parameter g_0

$$g_0 = Z_g\, g = g + \delta g \tag{39}$$

with renormalization constants Z_g different from 1 by a 1-loop term. The renormalized parameters g are finite and fixed by a set of renormalization conditions. The decomposition (39) is to a large extent arbitrary. Only the divergent parts are determined directly by the structure of the divergences of the one-loop amplitudes. The finite parts depend on the choice of the explicit renormalization conditions.

This procedure of parameter renormalization is sufficient to obtain finite S-matrix elements when wave function renormalization for external on-shell particles is included. Off-shell Green functions, however, are not finite by themselves. In order obtain finite propagators and vertices, also the bare fields in $\mathcal{L}$ have to be redefined in terms of renormalized fields by multiplicative renormalization

$$\phi_0 = Z_\phi^{1/2}\, \phi\,. \tag{40}$$

Expanding the renormalization constants according to

$$Z_i = 1 + \delta Z_i$$

the Lagrangian is split into a "renormalized" Lagrangian and a counter term Lagrangian

$$\mathcal{L}(\phi_0, g_0) = \mathcal{L}(Z_\phi^{1/2}\phi, Z_g g) = \mathcal{L}(\phi, g) + \delta\mathcal{L}(\phi, g, \delta Z_\phi, \delta g) \tag{41}$$

which renders the results for all Green functions in a given order finite.

The simplest way to obtain a set of finite Green functions is the "minimal subtraction scheme" [18] where (in dimensional regularization) the singular part of each divergent diagram is subtracted and the parameters are defined at an arbitrary mass scale μ. This scheme, with slight modifications, has been applied in QCD where due to the confinement of quarks and gluons there is no distinguished mass scale in the renormalization procedure.

The situation is different in QED and in the electroweak theory. There the classical Thomson scattering and the particle masses set natural scales where the parameters can be defined. In QED the favoured renormalization scheme is the on-shell scheme where $e = \sqrt{4\pi\alpha}$ and the electron, muon, ... masses are used as input parameters. The finite parts of the counter terms are fixed by the renormalization conditions that the fermion propagators have poles at their physical masses, and e becomes the $ee\gamma$ coupling constant in the Thomson limit of Compton scattering. The extraordinary meaning of the Thomson limit for the definition of the renormalized coupling constant is elucidated by the theorem that the exact Compton cross section at low energies becomes equal to the classical Thomson cross section. In particular this means that e resp. α is free of infrared corrections, and that its numerical value is independent of the order of perturbation theory, only determined by the accuracy of the experiment.

This feature of e is preserved in the electroweak theory. In the electroweak Standard Model a distinguished set for parameter renormalization is given in terms of e, M_Z, M_W, M_H, m_f with the masses of the corresponding particles. This electroweak on-shell scheme is the straight-forward extension of the familiar QED renormalization, first proposed by Ross and Taylor [19] and used in many practical applications [15, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. For stable particles, the masses are well defined quantities and can be measured with high accuracy. The masses of the W and Z bosons are related to the resonance peaks in cross sections where they are produced and hence can also be accurately determined. The masses of the Higgs boson and the top quark, as long as they are experimentally unknown, are treated as free input parameters. The light quark masses can only be considered as effective parameters. In the cases of practical interest they can be replaced in terms of directly measured quantities like the cross section for $e^+e^- \to$ hadrons.

The electroweak mixing angle is related to the vector boson masses in general by

$$\sin^2\theta_W = 1 - \frac{M_W^2}{\rho_0 M_Z^2} \tag{42}$$

where $\rho_0 \neq 1$ at the tree level in case of a Higgs system more complicated than with doublets only. We want to restrict our discussion of radiative corrections primarily to the minimal model with $\rho_0 = 1$. For $\rho \neq 1$ see section 8.2.

Instead of the set e, M_W, M_Z as basic free parameters one may alternatively use as basic parameters α, G_μ, M_Z [30] or α, G_μ, $\sin^2\theta_W$ with the mixing angle deduced from neutrino-electron scattering [31] or perform the loop calculations in the $\overline{MS}$ scheme [32, 33, 34, 35]. The so-called $*$-scheme [36, 37] is a different way of bookkeeping in terms of effective running couplings. We will discuss the various schemes in later course after the detailed discussion of the on-shell renormalization.

A full treatment of one-loop renormalization has to comprise also the unphysical sector. Since we are interested only in the calculation of physical amplitudes for light fermions at the one-loop level we drop the discussion of the unphysical sector.

3.2 *The renormalization transformation and counter terms*

Following the general principles discussed above we attach multiplicative renormalization constants to each free parameter and each symmetry multiplet of fields in the symmetric Lagrangian:

$$\begin{aligned}
W_\mu^a &\rightarrow \left(Z_2^W\right)^{1/2} W_\mu^a \\
B_\mu &\rightarrow \left(Z_2^B\right)^{1/2} B_\mu \\
\psi_j^L &\rightarrow \left(Z_L^j\right)^{1/2} \psi_j^L \\
\psi_{j\sigma}^R &\rightarrow \left(Z_R^{j\sigma}\right)^{1/2} \psi_{j\sigma}^R \\
\Phi &\rightarrow \left(Z^\Phi\right)^{1/2} \Phi \\
g_2 &\rightarrow Z_1^W \left(Z_2^W\right)^{-3/2} g_2 \\
g_1 &\rightarrow Z_1^B \left(Z_2^B\right)^{-3/2} g_1 \\
v &\rightarrow \left(Z^\Phi\right)^{1/2} (v - \delta v) \\
g_{j\sigma} &\rightarrow \left(Z^\Phi\right)^{-1/2} Z_1^{j\sigma} g_{j\sigma} \\
\mu^2 &\rightarrow \left(Z^\Phi\right)^{-1} (\mu^2 - \delta\mu^2) \\
\lambda &\rightarrow \left(Z^\Phi\right)^{-2} Z_\lambda \lambda
\end{aligned} \tag{43}$$

The r.h.s. represent the bare fields and parameters, the quantities without the Z-factors are the corresponding renormalized fields and parameters.

Field renormalization ensures that we end up with finite Green functions. The field renormalization in (43) is performed in a way that it respects the gauge symmetry by introducing the minimal number of field renormalization constants. Therefore also the counter term Lagrangian and the renormalized Green functions reflect the gauge symmetry. The price for this, however, is that not all residues of the propagators can be normalized to unity. As a consequence, any calculation with the renormalized Lagrangian will have to include finite multiplicative wave function renormalization factors for some of the external lines in S matrix elements.

It is of course possible to perform the renormalization in such a way that these finite corrections do not appear [23, 24, 25, 27]. But then the Lagrangian will contain many constants which have to be calculated in terms of the few fundamental parameters.

The independent renormalization of the Higgs vacuum expectation value v absorbs the linear term in the Higgs potential, which is induced by the appearance of tadpole diagrams in one-loop order, in such a way that the relation

$$v = \frac{2\mu}{\sqrt{\lambda}}$$

remains valid for the renormalized parameters with v being the minimum of the Higgs potential at the one-loop level. As a practical consequence of this tadpole renormalization, all tadpole graphs can be omitted in the renormalized amplitudes and Green functions. They are, however, necessary to make the mass counter terms gauge independent.

The systematic way for obtaining results for physical amplitudes in one-loop order is scheduled as follows: The expansion (41) yields the renormalized Lagrangian $\mathcal{L}$ which can now be re-parametrized in terms of the physical parameters (25) and the physical fields A_μ, Z_μ, $W_\mu^\pm$, H (also the unphysical Higgs field components $\phi^\pm$, χ, and the ghost fields u are present in the R_ξ gauge), and the counter term Lagrangian $\delta\mathcal{L}$. From $\delta\mathcal{L}$ the counter term Feynman rules are derived. After rewriting them in terms of (25) the counter term graphs have to be added to the 1-loop vertex functions calculated from $\mathcal{L}$. The renormalization constants in (43) are fixed afterwards by imposing the appropriate renormalization conditions. The results are finite Green functions in terms of the parameter set (25) from which the S matrix elements for all processes of interest can be obtained.

In order to perform mass and field renormalization we have to dress the propagators of the vector bosons and fermions by the self energies, i.e. the amputated 1-particle irreducible 2-point functions.

Vector boson self energies:
The self energies $\Sigma^{j\ell}$ enter the transverse components of the vector boson propagators $D_{\mu\nu}$ as follows ($V = \gamma,\ Z,\ W$):

$$\begin{aligned} D_{\mu\nu}^V(k) &= -ig_{\mu\nu}\left(\frac{1}{k^2-M_V^2} - \frac{1}{k^2-M_V^2}\,\Sigma^{VV}(k^2)\,\frac{1}{k^2-M_V^2}\right) \\ D_{\mu\nu}^{\gamma Z}(k) &= +ig_{\mu\nu}\,\frac{1}{k^2-M_Z^2}\,\Sigma^{\gamma Z}(k^2)\,\frac{1}{k^2}. \end{aligned} \tag{44}$$

We can drop the longitudinal components $\sim k_\mu k_\nu$ since they only yield terms which are suppressed by m_f^2/M_V^2 in physical amplitudes.

The Σ's represent the sum of all contributing one-loop diagrams. The corresponding renormalized self energies are obtained by adding the counter terms derived from $\delta\mathcal{L}$. It is convenient to introduce the following linear combinations of the SU(2) and the U(1) field renormalization constants $\delta Z_2^{W,B}$ and the coupling renormalization constants $\delta Z_1^{W,B}$ $(i = 1, 2)$:

$$\begin{pmatrix} \delta Z_i^\gamma \\ \delta Z_i^Z \end{pmatrix} = \begin{pmatrix} s_W^2 & c_W^2 \\ c_W^2 & s_W^2 \end{pmatrix} \begin{pmatrix} \delta Z_i^W \\ \delta Z_i^B \end{pmatrix}, \tag{45}$$

with the abbreviations

$$s_W^2 = \sin^2\theta_W, \quad c_W^2 = \cos^2\theta_W\,. \tag{46}$$

Using the notation (45) we can write down the renormalized self energies as follows (all renormalized quantities are denoted by the same symbols as the corresponding unrenormalized ones in connection with a superscript $\hat{}$):

$$\begin{aligned} \hat{\Sigma}^{\gamma\gamma}(k^2) &= \Sigma^{\gamma\gamma}(k^2) + \delta Z_2^\gamma\, k^2 \\ \hat{\Sigma}^{ZZ}(k^2) &= \Sigma^{ZZ}(k^2) - \delta M_Z^2 + \delta Z_2^Z\,(k^2 - M_Z^2) \\ \hat{\Sigma}^{WW}(k^2) &= \Sigma^{WW}(k^2) - \delta M_W^2 + \delta Z_2^W\,(k^2 - M_W^2) \\ \hat{\Sigma}^{\gamma Z}(k^2) &= \Sigma^{\gamma Z}(k^2) - \delta Z_2^{\gamma Z}\, k^2 + (\delta Z_1^{\gamma Z} - \delta Z_2^{\gamma Z})\, M_Z^2\,. \end{aligned} \tag{47}$$

In the last line the combinations $(i = 1, 2)$

$$\delta Z_i^{\gamma Z} = \frac{c_W s_W}{c_W^2 - s_W^2}\,(\delta Z_i^Z - \delta Z_i^\gamma) \tag{48}$$

have been introduced.

The mass counter terms $\delta M_{W,Z}^2$ following from (19) and (43) are related to the fundamental renormalization constants by

$$\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} = \frac{s_W}{c_W}\,(3\,\delta Z_2^{\gamma Z} - 2\,\delta Z_1^{\gamma Z})\,. \tag{49}$$

This relation allows us to express finally the $\delta Z_i^{Z,W}$ in terms of the unrenormalized on-shell vector boson self energies.

Fermion self energies:
The fermion self energy Σ^f is related to the fermion propagator in the following way:

$$S_F^f(k) = \frac{i}{\not k - m_f} - \frac{i}{\not k - m_f}\,\Sigma^f(k)\,\frac{1}{\not k - m_f}. \tag{50}$$

Σ^f can be decomposed according to

$$\begin{aligned} \Sigma^f(k) &= \not k\,\Sigma_V^f(k^2) + \not k\,\gamma_5\,\Sigma_A^f(k^2) + m_f\,\Sigma_S^f(k^2) \\ &= \not k\,\frac{1-\gamma_5}{2}\,\Sigma_L^f(k^2) + \not k\,\frac{1+\gamma_5}{2}\,\Sigma_R^f(k^2) + m_f\,\Sigma_S^f(k^2) \end{aligned} \tag{51}$$

with scalar functions $\Sigma^f_{V,A,S}$ resp. $\Sigma^f_{L,R,S}$ related via

$$\Sigma_L = \Sigma_V - \Sigma_A, \quad \Sigma_R = \Sigma_V + \Sigma_A .$$

By adding the counter terms as derived from our renormalization transformation (43) obtain the renormalized fermion self energies:

$$\begin{aligned} \hat{\Sigma}^f(k) &= \not{k}\left(\Sigma^f_V(k^2) + \delta Z^f_V\right) + \not{k}\gamma_5\left(\Sigma^f_A(k^2) - \delta Z^f_A\right) \\ &\quad + m_f\left(\Sigma^f_S(k^2) - \delta Z^f_V - \frac{\delta m_f}{m_f}\right) \end{aligned} \tag{52}$$

with

$$\delta Z^f_V = \frac{\delta Z_L + \delta Z^f_R}{2}, \quad \delta Z^f_A = \frac{\delta Z_L - \delta Z^f_R}{2} . \tag{53}$$

δZ_L is the left-handed renormalization constant for the whole doublet; therefore not all of the $\delta Z^f_{V,A}$ are independent for the members of a family. We have dropped the family index in the formulae.

The mass renormalization

$$\frac{\delta m_f}{m_f} = \delta Z^f_1 - \frac{\delta v}{v} \tag{54}$$

contains, besides δv, the Yukawa coupling renormalization constant δZ^f_1. Consequently, fermion mass renormalization fixes the renormalization of the Yukawa couplings which is of interest e.g. for the discussion of the fermionic Higgs boson decays at the one-loop level [38].

Vertex corrections

For coupling constant renormalization and for dressing the fermion gauge boson vertices we have to add the counter terms following from (43) in order to obtain the renormalized vertices. With the coupling constants v_f, a_f of the fermion f to the Z specified in (36) and (37) we get the renormalized electromagnetic vertex as

$$\begin{aligned} \hat{\Gamma}^{\gamma ff}_\mu &= \Gamma^{\gamma ff}_\mu - i\,e\,Q_f\,\gamma_\mu\,(\delta Z^\gamma_1 - \delta Z^\gamma_2 + \delta Z^f_V - \delta Z^f_A\,\gamma_5) \\ &\quad - i\,\frac{e}{2 s_W c_W}\,\gamma_\mu(v_f - a_f\gamma_5)\,(\delta Z^{\gamma Z}_1 - \delta Z^{\gamma Z}_2) \end{aligned} \tag{55}$$

with the unrenormalized vertex

$$\Gamma^{\gamma ff}_\mu = -\,i\,e\,Q_f\,\gamma_\mu + i\,e\,\Lambda^{\gamma f}_\mu . \tag{56}$$

$\Lambda^{\gamma f}_\mu$ denote the one-loop vertex corrections. For light on-shell fermions ($f \neq b, t$)) with momenta p, p' and $k^2 = (p - p')^2 \gg m_f^2$ they essentially consist of vector and axial vector form factors only:

$$\Lambda^{\gamma f}_\mu = i\,e\,\gamma_\mu\,[\Lambda^{\gamma f}_V(k^2) - \gamma_5\,\Lambda^{\gamma f}_A(k^2)] . \tag{57}$$

The renormalized weak neutral current vertex has the form

$$
\begin{aligned}
\hat{\Gamma}^{Zff}_\mu &= \Gamma^{Zff}_\mu + i\,\frac{e}{2s_W c_W}\,\gamma_\mu(v_f - a_f\gamma_5)\,(\delta Z_1^Z - \delta Z_2^Z) \\
&\quad + i\,e\,Q_f\,\gamma_\mu\,(\delta Z_1^{\gamma Z} - \delta Z_2^{\gamma Z}) \\
&\quad + i\,\frac{e}{2s_W c_W}\,\gamma_\mu\,(v_f\delta Z_V^f + a_f\delta Z_A^f) \\
&\quad - i\,\frac{e}{2s_W c_W}\,\gamma_\mu\gamma_5\,(v_f\delta Z_A^f + a_f\delta Z_V^f)\,. \qquad (58)
\end{aligned}
$$

The renormalized charged current vertex reads:

$$
\hat{\Gamma}^{CC}_\mu = \Gamma^{CC}_\mu + i\,\frac{e}{2\sqrt{2}s_W}\,\gamma_\mu(1-\gamma_5)\,(1 + \delta Z_1^W - \delta Z_2^W + \delta Z_L) \qquad (59)
$$

with δZ_L for the corresponding lepton or quark doublet. Γ_μ always denote the unrenormalized vertices:

$$
\begin{aligned}
\Gamma^{Zff}_\mu &= i\,\frac{e}{2s_W c_W}\,\gamma_\mu(v_f - a_f\gamma_5) + i\,e\,\Lambda^{Zf}_\mu \\
\Gamma^{CC}_\mu &= i\,\frac{e}{2\sqrt{2}s_W}\gamma_\mu(1-\gamma_5) + i\,e\,\Lambda^{CC}_\mu\,. \qquad (60)
\end{aligned}
$$

The one-loop vertex corrections Λ_μ have a similar decomposition into form factors as given above for the photon case.

3.3 *On-shell renormalization conditions and renormalization constants*

The renormalization conditions can be separated into two classes: the on-shell subtraction of the self energies which makes the particle content of the theory evident, and the generalization of the QED charge renormalization. Since we have introduced more renormalization constants than physical parameters we are free to fix the supernumerary ones by the requirement of residue$=1$ for a corresponding number of propagators. In order to be as close as possible to the common QED renormalization these residue conditions are imposed on the photon and the charged lepton propagators.

The on-shell subtraction conditions can be written in the following way:

$$
\mathrm{Re}\,\hat{\Sigma}^{WW}(M_W^2) = \mathrm{Re}\,\hat{\Sigma}^{ZZ}(M_Z^2) = \mathrm{Re}\,\hat{\Sigma}^f(\not{p} = m_f) = 0\,. \qquad (61)
$$

The "QED-like" conditions read explicitly:

$$
\hat{\Gamma}^{\gamma ee}_\mu(k^2 = 0, \not{p} = \not{q} = m_e) = ie\gamma_\mu
$$

$$\begin{aligned}
\hat{\Sigma}^{\gamma Z}(0) &= 0 \\
\frac{\partial \hat{\Sigma}^{\gamma\gamma}}{\partial k^2}(0) &= 0 \\
\lim_{\not{k}\to m_-} \frac{1}{\not{k} - m_-}\, \hat{\Sigma}^f(k)\, u_-(k) &= 0
\end{aligned} \tag{62}$$

if u_- is the wave function for the $I_3 = -1/2$ particle.

The last condition is formulated for charged leptons and quarks with $I_3 = -1/2$. It means a condition for the left and right handed fermion field renormalization constants Z_L, Z_R^-. In the case of leptons Z_L also determines the neutrino field renormalization. For the $I_3 = +1/2$ quarks an additional Z_R^+ for the right handed fields is at our disposal. This constant can be adjusted in a way that the renormalized left and right handed parts of the up-type propagators have equal residues at $k^2 = m_+^2$ (but $\neq 1$).

The solution of the system (61)and (62) yields all those renormalization constants which we need for the vector boson propagators, the fermion - gauge boson vertex corrections, and the fermion wave function renormalization. We write them down in terms of the unrenormalized expressions.

The mass counter terms for the W and Z self energies follow immediately from the unrenormalized on-shell values by means of Eq. (47) and (61):

$$\begin{aligned}
\delta M_W^2 &= \mathrm{Re}\, \Sigma^{WW}(M_W^2) \\
\delta M_Z^2 &= \mathrm{Re}\, \Sigma^{ZZ}(M_Z^2)\,.
\end{aligned} \tag{63}$$

Their dependence on the $Z_i^{\gamma,Z}$ $(i = 1,2)$ in (49) together with the set of equations (62) yields:

$$\begin{aligned}
\delta Z_2^\gamma &= -\Pi^\gamma(0) \equiv -\frac{\partial \Sigma^\gamma}{\partial k^2}(0) \\
\delta Z_1^\gamma &= -\Pi^\gamma(0) - \frac{s_W}{c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} \\
\delta Z_2^Z &= -\Pi^\gamma(0) - 2\frac{c_W^2 - s_W^2}{s_W c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} + \frac{c_W^2 - s_W^2}{s_W^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) \\
\delta Z_1^Z &= -\Pi^\gamma(0) - \frac{3c_W^2 - 2s_W^2}{s_W c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} + \frac{c_W^2 - s_W^2}{s_W^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) \\
\delta Z_2^W &= -\Pi^\gamma(0) - 2\frac{c_W}{s_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} + \frac{c_W^2}{s_W^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) \\
\delta Z_1^W &= -\Pi^\gamma(0) - \frac{3 - 2s_W^2}{s_W c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} + \frac{c_W^2}{s_W^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right).
\end{aligned} \tag{64}$$

The last two constants δZ_i^W are not independent but are linear combinations of δZ_i^γ and δZ_i^Z. They are given here for completeness.

The mass renormalization term for a massive fermion f is determined by

$$\frac{\delta m_f}{m_f} = \Sigma_V^f(m_f^2) + \Sigma_S^f(m_f^2)\,. \tag{65}$$

The doublet field renormalization constants δZ_L and the singlet renormalization constants δZ_R^- in the $I_3 = -1/2$ states follow from (62) to be:

$$\begin{aligned} \delta Z_L &= -\Sigma_L(m_-^2) - m_-^2\left[\Sigma_L'(m_-^2) + \Sigma_R'(m_-^2) + 2\Sigma_S'(m_-^2)\right] \\ \delta Z_R^- &= -\Sigma_R(m_-^2) - m_-^2\left[\Sigma_L'(m_-^2) + \Sigma_R'(m_-^2) + 2\Sigma_S'(m_-^2)\right] \end{aligned} \tag{66}$$

The $\Sigma_{L,..}$ are the invariant functions in Eq. (51), and $\Sigma_{L,..}'$ denotes the derivative

$$\Sigma_{L,..}'(k^2) = \frac{\partial \Sigma_{L,..}}{\partial k^2}.$$

By means of (53) one can rewrite (66) as follows:

$$\begin{aligned} \delta Z_V^- &= -\Sigma_V(m_-^2) - 2m_-^2\left[\Sigma_V'(m_-^2) + \Sigma_S'(m_-^2)\right] \\ \delta Z_A^- &= +\Sigma_A(m_-^2). \end{aligned} \tag{67}$$

In the case of leptons δZ_L renormalizes simultaneously the neutrino propagator with the consequence that its residue is different from 1 by the finite amount

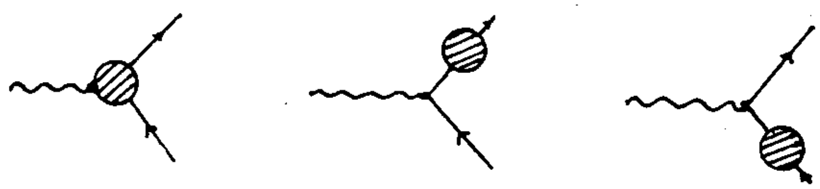
$$\hat{\Pi}^\nu = -\Sigma_L^\nu(0) - \delta Z_L. \tag{68}$$

Therefore, in external ν lines a finite wave function renormalization factor

$$1 - \frac{1}{2}\left[\Sigma_L^\nu(0) + \delta Z_L\right] \tag{69}$$

has to be inserted.

For the right handed u-type quarks an additional condition has to be imposed in order to fix δZ_R^+. We will treat the $I_3 = +1/2$ quarks in a way that the residues for their left and right handed propagators become equal. This looks somewhat arbitrary. In S matrix elements, however, in the sum



this unsymmetric treatment is compensated by the corresponding renormalized 3-point vertices.

We have skipped the Higgs mass on-shell renormalization since it is not needed for our purpose. The on-shell subtraction for the Higgs self energy together with the tadpole conditions fix also the scalar sector at the one-loop level.

Charge renormalization:

It is instructive to have a look at the direct counter term δe for the renormalization of the electric charge, which can easily be derived from the basic renormalization constants given above in Eq. (64). Making use of the definition (21) and the renormalization transformation (RT) in Eq. (43) we obtain

$$e^2 = \frac{g_1^2 g_2^2}{g_1^2+g_2^2} \xrightarrow{RT} \frac{g_1^2 g_2^2}{g_1^2+g_2^2}\,(1+2\delta Z_1^\gamma - 3\delta Z_2^\gamma) \equiv e^2\left(1+2\frac{\delta e}{e}\right) \tag{70}$$

where

$$\frac{\delta e}{e} = \delta Z_1^\gamma - \frac{3}{2}\delta Z_2^\gamma = \frac{1}{2}\Pi^\gamma(0) - \frac{s_W}{c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2}\,. \tag{71}$$

This demonstrates that δe, although fixed in terms of the electron specific condition (62), is fermion independent. The first of the two universal terms in Eq. (71) is the photon vacuum polarization, as in QED (but including also bosonic loops), the second term contains the mixing between the photon and the Z boson. As we will see in the next section, the fermion loops vanish for $k^2 = 0$. Only the non-abelian bosonic loops yield $\Sigma^{\gamma Z}(0) \neq 0$.

The reason for the universality of δe is the $U(1)_Y$ Ward identity, which is formally identical to the QED Ward identity [39], yielding the following relation between the field and coupling renormalization constants for the U(1) part:

$$\delta Z_1^B = \delta Z_2^B\,.$$

Exploiting this relation together with the second condition of (62) allows us to write down the following identity:

$$\delta Z_1^\gamma - \delta Z_2^\gamma = -\frac{s_W}{c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} \tag{72}$$

which immediately leads to (71).

The charge renormalization condition in (62) is an explicit condition for the vector part of the electron vertex:

$$\Lambda_V^{\gamma e}(0) + \delta Z_V^e + \delta Z_1^\gamma - \delta Z_2^\gamma + \frac{4s_W^2-1}{4s_W c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} = 0\,. \tag{73}$$

The identity (72) shows that the fermion specific part of the vertex corrections and of the field renormalization cancel in the combination entering the condition (73):

$$\Lambda_V^{\gamma e}(0) + \delta Z_V^e = \frac{1}{4s_W c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2}\,. \tag{74}$$

Another consequence of the Ward identity is the absence of an electromagnetic axial vector coupling for real photons. The renormalized electromagnetic axial vector vertex in Eq. (55) is finite and vanishes for $k^2 \to 0$. Simultaneously, this also holds for the electromagnetic form factor of the neutrino.

Renormalization of $\sin^2\theta_W$:

The counter term for the electroweak mixing angle can be derived in a similar way as done for δe on the basis of the definition (22), the transformation (43), and the relation (49):

$$s_W^2 = \frac{g_1^2}{g_1^2+g_2^2} \xrightarrow{RT} s_W^2 + c_W^2\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) \equiv s_W^2 + \delta s_W^2 \tag{75}$$

with

$$s_W^2 = 1 - \frac{M_W^2}{M_Z^2}\,.$$

This becomes obvious also in a more direct way from expanding the exact relation between the bare quantities up to the 1-loop order:

$$\begin{aligned} s_0^2 = 1 - \frac{M_W^{0\,2}}{M_Z^{0\,2}} &= 1 - \frac{M_W^2+\delta M_W^2}{M_Z^2+\delta M_Z^2} \\ &= 1 - \frac{M_W^2}{M_Z^2} + \frac{M_W^2}{M_Z^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right). \end{aligned} \tag{76}$$

Concluding this discussion we summarize the principal structure of electroweak calculations:

- The treel level Lagrangian $\mathcal{L}(e, M_W, M_Z, \ldots)$ is sufficent for lowest order calculations and the parameters can be identified with the physical parameters.
- For higher order calculations, $\mathcal{L}$ has to be considered as the bare Lagrangian of the theory $\mathcal{L}(e_0, M_W^0, M_Z^0, \ldots)$ with bare parameters which are related to the physical ones by

  $$e_0 = e + \delta e, \quad M_W^{0\,2} = M_W^2 + \delta M_W^2, \quad M_Z^{0\,2} = M_Z^2 + \delta M_Z^2\,.$$

  The counterterms are fixed in terms of a specific set of 1-loop self energies via Eq.s (63), (71), (76). For any 4-fermion process the S-matrix element with the corresponding loop diagrams and the counter terms is finite after external wave function renormalization.
- When field renormalization is performed, also the individual self energies and vertex corrections are finite.

4 Calculation of one-loop integrals

In this section we provide the technical details for the calculation of radiative corrections for electroweak precision observables. The methods used are essentially based on the work of [20] and [40].

4.1 *Dimensional regularization*

The diagrams with closed loops occuring in higher order perturbation theory involve integrals over the loop momentum. These integrals are in general divergent for large integration momenta (UV divergence). For this reason we need a regularization, which is a procedure to redefine the integrals in such a way that they become finite and mathematically well-defined objects. The widely used regularization procedure for gauge theories is that of dimensional regularization [41], which is Lorentz and gauge invariant: replace the dimension 4 by a lower dimension D where the integrals are convergent:

$$\int \frac{d^4k}{(2\pi)^4} \rightarrow \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} \tag{77}$$

An (arbitrary) mass parameter μ has been introduced in order to keep the dimensions of the coupling constants in front of the integrals independent of D. After renormalization the results for physical quantities are finite in the limit $D \rightarrow 4$.

The metric tensor in D dimensions has the property

$$g^\mu_\mu = g_{\mu\nu} g^{\nu\mu} = \mathrm{Tr}(1) = D\,. \tag{78}$$

The Dirac algebra in D dimensions

$$\{\gamma_\mu, \gamma_\nu\} = 2\, g_{\mu\nu}\, \mathbf{1} \tag{79}$$

has the consequences

$$\begin{aligned}
\gamma_\mu \gamma^\mu &= D\,\mathbf{1} \\
\gamma_\rho \gamma_\mu \gamma^\rho &= (2-D)\,\gamma_\mu \\
\gamma_\rho \gamma_\mu \gamma_\nu \gamma^\rho &= 4 g_{\mu\nu}\,\mathbf{1} - (4-D)\,\gamma_\mu\gamma_\nu \\
\gamma_\rho \gamma_\mu \gamma_\nu \gamma_\sigma \gamma^\rho &= -2\,\gamma_\sigma\gamma_\nu\gamma_\mu + (4-D)\,\gamma_\mu\gamma_\nu\gamma_\sigma
\end{aligned} \tag{80}$$

A consistent treatment of γ_5 in D dimensions is more subtle [42]. In theories which are anomaly free like the Standard Model we can use γ_5 as anticommuting with γ_μ:

$$\{\gamma_\mu, \gamma_5\} = 0\,. \tag{81}$$

4.2 *One- and two-point integrals*

In the calculation of self energy diagrams the following types of one-loop integrals appear:

1-point integral:

$$\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{k^2-m^2} \quad =: \quad \frac{i}{16\pi^2}\,A(m) \tag{82}$$

2-point integrals:

$$\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{[k^2-m_1^2][(k+q)^2-m_2^2]} \quad =: \quad \frac{i}{16\pi^2}\,B_0(q^2,m_1,m_2) \tag{83}$$

$$\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{k_\mu;\,k_\mu k_\nu}{[k^2-m_1^2][(k+q)^2-m_2^2]} \quad =: \quad \frac{i}{16\pi^2}\,B_{\mu;\,\mu\nu}(q^2,m_1,m_2)\,. \tag{84}$$

The vector and tensor integrals B_μ, $B_{\mu\nu}$ can be expanded into Lorentz covariants and scalar coefficients:

$$\begin{aligned} B_\mu &= q_\mu\,B_1(q^2,m_1,m_2) \\ B_{\mu\nu} &= g_{\mu\nu}B_{22}(q^2,m_1,m_2) + q_\mu q_\nu B_{21}(q^2,m_1,m_2)\,. \end{aligned} \tag{85}$$

The coefficient functions can be obtained algebraically from the scalar 1- and 2-point integrals A and B_0. Contracting (84) with q^μ, $g^{\mu\nu}$ and $q^\mu q^\nu$ yields:

$$\begin{aligned} \int\frac{kq}{[k^2-m_1^2][(k+q)^2-m_2^2]} &= \frac{i}{16\pi^2}\,q^2B_1 \\ \int\frac{k^2}{[k^2-m_1^2][(k+q)^2-m_2^2]} &= \frac{i}{16\pi^2}\left(DB_{22}+q^2B_{21}\right) \\ \int\frac{(kq)^2}{[k^2-m_1^2][(k+q)^2-m_2^2]} &= \frac{i}{16\pi^2}\left(q^2B_{22}+q^4B_{21}\right)\,. \end{aligned} \tag{86}$$

Solving these equations and making use of the decompositions

$$\int\frac{k^2}{[k^2-m_1^2][(k+q)^2-m_2^2]} = \int\frac{1}{k^2-m_2^2}+m_1^2\int\frac{1}{[k^2-m_1^2][(k+q)^2-m_2^2]}$$

$$\int \frac{kq}{[k^2 - m_1^2][(k+q)^2 - m_2^2]} = \frac{1}{2}\int \frac{1}{k^2 - m_1^2} - \frac{1}{2}\int \frac{1}{k^2 - m_2^2} + \frac{m_2^2 - m_1^2 - q^2}{2}\int \frac{1}{[k^2 - m_1^2][(k+q)^2 - m_2^2]}$$

$$\int \frac{(kq)^2}{[k^2 - m_1^2][(k+q)^2 - m_2^2]} = \frac{1}{2}\int \frac{kq}{k^2 - m_1^2} - \frac{1}{2}\int \frac{kq}{(k+q)^2 - m_2^2} + \frac{m_2^2 - m_1^2 - q^2}{2}\int \frac{kq}{[k^2 - m_1^2][(k+q)^2 - m_2^2]}$$

and of the definition (82,83) we obtain:

$$\begin{aligned}
B_1(q^2, m_1, m_2) &= \frac{1}{2q^2}\left[A(m_1) - A(m_2) + (m_1^2 - m_2^2 - q^2)B_0(q^2, m_1, m_2)\right] \\
B_{22}(q^2, m_1, m_2) &= \frac{1}{6}\left[A(m_2) + 2m_1^2 B_0(q^2, m_1, m_2)\right. \\
&\quad \left. +(q^2 + m_1^2 - m_2^2)B_1(q^2, m_1, m_2) + m_1^2 + m_2^2 - \frac{q^2}{3}\right] \\
B_{21}(q^2, m_1, m_2) &= \frac{1}{3q^2}\left[A(m_2) - m_1^2 B_0(q^2, m_1, m_2)\right. \\
&\quad \left. -2(q^2 + m_1^2 - m_2^2)B_1(q^2, m_1, m_2) - \frac{m_1^2 + m_2^2}{2} + \frac{q^2}{6}\right] . \qquad (87)
\end{aligned}$$

Finally we have to calculate the scalar integrals A and B_0. With help of the Feynman parametrization

$$\frac{1}{ab} = \int_0^1 dx \, \frac{1}{[ax + b(1-x)]^2}$$

and after a shift in the k-variable, B_0 can be written in the form

$$\frac{i}{16\pi^2} B_0(q^2, m_1, m_2) = \int_0^1 dx \, \frac{\mu^{4-D}}{(2\pi)^D} \int \frac{d^D k}{[k^2 - x^2 q^2 + x(q^2 + m_1^2 - m_2^2) - m_1^2]^2} . \qquad (88)$$

The advantage of this parametrization is a simpler k-integration where the integrand is only a function of $k^2 = (k^0)^2 - \vec{k}^2$. In order to transform it into a Euclidean integral we perform the substitution [2]

$$k^0 = i\, k_E^0, \;\; \vec{k} = \vec{k_E}, \;\; d^D k = i\, d^D k_E$$

[2]The $i\epsilon$-prescription in the masses ensures that this is compatible with the pole structure of the integrand.

where the new integration momentum k_E has a definite metric:

$$k^2 = -k_E^2, \quad k_E^2 = (k_E^0)^2 + \cdots + (k_E^{D-1})^2 .$$

This leads us to a Euclidean integral over k_E:

$$\frac{i}{16\pi^2} B_0 = i \int_0^1 dx \frac{\mu^{4-D}}{(2\pi)^D} \int \frac{d^D k_E}{(k_E^2 + Q)^2} \tag{89}$$

where

$$Q = x^2 q^2 - x(q^2 + m_1^2 - m_2^2) + m_1^2 - i\varepsilon \tag{90}$$

is a constant with respect to the k_E-integration.

Also the 1-point integral A of (82) can be transformed into a Euclidean integral:

$$\frac{i}{16\pi^2} A(m) = -i \frac{\mu^{4-D}}{(2\pi)^D} \int \frac{d^D k_E}{k_E^2 + m^2} . \tag{91}$$

Both k_E- integrals are of the general type

$$\int \frac{d^D k_E}{(k_E^2 + L)^n}$$

of rotational invariant integrals in a D-dimensional Euclidean space. They can be evaluated in D-dimensional polar coordinates ($k_E^2 = R$)

$$\int \frac{d^D k_E}{(k_E^2 + L)^n} = \frac{1}{2} \int d\Omega_D \int_0^\infty dR\, R^{\frac{D}{2}-1} \frac{1}{(R+L)^n} ,$$

yielding

$$\frac{\mu^{4-D}}{(2\pi)^D} \int \frac{d^D k_E}{(k_E^2 + L)^n} = \frac{\mu^{4-D}}{(4\pi)^{D/2}} \cdot \frac{\Gamma(n - \frac{D}{2})}{\Gamma(n)} \cdot L^{-n+\frac{D}{2}} . \tag{92}$$

The singularities of our initially 4-dimensional integrals are now recovered as poles of the Γ-function for $D = 4$ and values $n \leq 2$.

Although the l.h.s. of Eq. (92) as a D-dimensional integral is sensible only for integer values of D, the r.h.s. has an analytic continuation in the variable D: it is well defined for all complex values D with $n - \frac{D}{2} \neq 0, -1, -2, \ldots$, in particular for

$$D = 4 - \epsilon \quad \text{with } \epsilon > 0 .$$

For physical reasons we are interested in the vicinity of $D = 4$. Hence we consider the limiting case $\epsilon \to 0$ and perform an expansion around $D = 4$ in powers of ϵ. For this task we need the following properties of the Γ-function at $x \to 0$:

$$\Gamma(x) = \frac{1}{x} - \gamma + O(x) ,$$

$$\Gamma(-1 + x) = -\frac{1}{x} + \gamma - 1 + O(x) \tag{93}$$

with

$$\gamma = -\Gamma'(1) = 0.577\ldots$$

known as Euler's constant.

$n = 1$:
Combining (91) and (92) we obtain the scalar 1-point integral for $D = 4 - \epsilon$:

$$\begin{aligned} A(m) &= -\frac{\mu^\epsilon}{(4\pi)^{-\epsilon/2}} \cdot \frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(1)} \cdot \left(m^2\right)^{1-\epsilon/2} \\ &= m^2 \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\frac{m^2}{\mu^2} + 1\right) + O(\epsilon) \\ &\equiv m^2 \left(\Delta - \log\frac{m^2}{\mu^2} + 1\right) + O(\epsilon) \end{aligned} \tag{94}$$

Here we have introduced the abbreviation for the singular part

$$\Delta = \frac{2}{\epsilon} - \gamma + \log 4\pi\,. \tag{95}$$

$n = 2$:
For the scalar 2-point integral B_0 we evaluate the integrand of the x-integration in Eq. (89) with help of Eq. (92) as follows:

$$\begin{aligned} \frac{\mu^\epsilon}{(4\pi)^{2-\epsilon/2}} \cdot \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)} \cdot Q^{-\epsilon/2} &= \frac{1}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\frac{Q}{\mu^2}\right) + O(\epsilon) \\ &= \frac{1}{16\pi^2}\left(\Delta - \log\frac{Q}{\mu^2}\right) + O(\epsilon)\,. \end{aligned} \tag{96}$$

Since the $O(\epsilon)$ terms vanish in the limit $\epsilon \to 0$ we skip them in the following formulae. Insertion into Eq. (89) with Q from Eq. (90) yields:

$$B_0(q^2, m_1, m_2) = \Delta - \int_0^1 dx\, \log\frac{x^2 q^2 - x(q^2 + m_1^2 - m_2^2) + m_1^2 - i\varepsilon}{\mu^2} \tag{97}$$

The explicit analytic result can be found in [15].

4.3 *Three-point integrals*

In the calculation of vertex corrections the following scalar, vector, and tensor 3-point integrals occur:

$$\mu^{4-D}\int \frac{d^D k}{(2\pi)^D}\, \frac{1;\, k_\mu;\, k_\mu k_\nu}{[k^2 - m_1^2][(k+p_1)^2 - m_2^2][(k+p_1+p_2)^2 - m_3^2]} = \frac{i}{16\pi^2}\, C_{0;\,\mu;\,\mu\nu}\,. \tag{98}$$

Expanding into Lorentz covariants

$$\begin{aligned} C^\mu &= p_1^\mu C_{11} + p_2^\mu C_{12} \\ C^{\mu\nu} &= g^{\mu\nu} C_{20} + p_1^\mu p_1^\nu C_{21} + p_2^\mu p_2^\nu C_{22} \\ &\quad + (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) C_{23} \end{aligned} \tag{99}$$

and performing all possible tensor contractions yields the coefficient functions in terms of the scalar 3-point integral C_0 and the 2-point integrals. The vector coefficients read:

$$\begin{aligned} C_{11} &= [p_2^2 R_1 - (p_1 p_2) R_2]/\kappa \\ C_{12} &= [-(p_1 p_2) R_1 + p_1^2 R_2]/\kappa \end{aligned} \tag{100}$$

where

$$\kappa = p_1^2 p_2^2 - (p_1 p_2)^2$$

and

$$\begin{aligned} R_1 &= [B_0(p_3^2, m_1, m_3) - B_0(p_2^2, m_2, m_3) - (p_1^2 + m_1^2 - m_2^2) C_0]/2 \\ R_2 &= [B_0(p_1^2, m_1, m_2) - B_0(p_3^2, m_1, m_3) + (p_1^2 - p_3^2 - m_2^2 + m_3^2) C_0]/2 \end{aligned} \tag{101}$$

with the notation

$$p_3^2 = (p_1 + p_2)^2 \,.$$

The tensor coefficients are given by

$$\begin{aligned} C_{20} &= [B_0(p_2^2, m_2, m_3) + r_1 C_{11} + r_2 C_{12} + 2m_1^2 C_0 + 1]/4 \\ C_{21} &= [p_2^2 R_3 - (p_1 p_2) R_5]/\kappa \\ C_{23} &= [-(p_1 p_2) R_3 + p_1^2 R_5]/\kappa \\ C_{22} &= [-(p_1 p_2) R_4 + p_1^2 R_6]/\kappa \end{aligned} \tag{102}$$

with

$$r_1 = p_1^2 + m_1^2 - m_2^2, \quad r_2 = p_3^2 - p_1^2 + m_2^2 - m_3^2$$

and

$$\begin{aligned} R_3 &= -C_{20} - [r_1 C_{11} - B_1(p_3^2, m_1, m_3) - B_0(p_2^2, m_2, m_3)]/2 \\ R_5 &= -[r_2 C_{11} - B_1(p_1^2, m_1, m_2) + B_1(p_3^2, m_1, m_3)]/2 \\ R_4 &= -[r_1 C_{12} - B_1(p_3^2, m_1, m_3) + B_1(p_2^2, m_2, m_3)]/2 \\ R_6 &= -C_{20} - [r_2 C_{12} + B_1(p_3^2, m_1, m_3)]/2 \end{aligned} \tag{103}$$

The genuine new element in the expressions above is the scalar 3-point integral

$$\frac{i}{16\pi^2} C_0(p_1, p_2; m_1, m_2, m_3) = \int \frac{1}{[k^2 - m_1^2][(k+p_1)^2 - m_2^2][(k+p_1+p_2)^2 - m_3^2]} \,.$$

After Feynman parametrization

$$\frac{1}{D_1 D_2 D_3} = \int_0^1 dx \int_0^1 dy \frac{1}{\{(1-x)D_1 + x[yD_2 + (1-y)D_3]\}^3}$$

and Wick rotation the momentum integration can be performed applying Eq. (92) for $n = 3$ and $D = 4$. The result is a 2-parameter integral for C_0:

$$C_0(p_1, p_2; m_1, m_2, m_3) = -\int_0^1 dx \int_0^x dy \frac{1}{ax^2 + by^2 + cxy + dx + ey + f} \tag{104}$$

with

$$\begin{aligned} &a = p_3^2, \quad b = p_2^2, \quad c = p_1^2 - p_2^2 - p_3^2, \\ &d = m_3^2 - m_1^2 - p_3^2, \\ &e = m_2^2 - m_1^2 + p_3^2 - p_1^2, \\ &f = m_1^2 - i\varepsilon . \end{aligned} \tag{105}$$

For real solutions α of the quadratic equation

$$b\alpha^2 + c\alpha + a = 0$$

the integral can be expressed as a sum over dilogarithms:

$$C_0 = \frac{1}{c + 2\alpha b} \sum_{l=1}^{3} \sum_{j=1}^{2} (-1)^l \left\{ \mathrm{Li}_2\left(\frac{x_l}{x_l - y_{lj}}\right) - \mathrm{Li}_2\left(\frac{x_l - 1}{x_l - y_{lj}}\right) \right\} \tag{106}$$

together with

$$\begin{aligned} x_1 &= -\frac{d + 2a + c\alpha}{c + 2\alpha b}, \\ x_2 &= -\frac{d}{(1-\alpha)(c + 2\alpha b)}, \\ x_3 &= \frac{d}{\alpha(c + 2\alpha b)}, \\ y_{1j} &= \frac{-c \pm \sqrt{c^2 - 4b(a + d + f)}}{2b}, \\ y_{2j} = y_{3j} &= \frac{-d \pm \sqrt{d^2 - 4f(a + b + c)}}{2a}. \end{aligned} \tag{107}$$

The condition for α being real is always fulfilled for vertices with two particles on-shell, in particular for 2-particle decay and scattering processes.

For the special situation of vertex corrections for the fermion-gauge boson vertices with light fermions the expressions become considerably simpler. The analytic results for the Zff vertices will be given in the next chapter on e^+e^- annihilation. For low energy processes, like muon decay or neutrino scattering, where the external momenta can be neglected in view of the internal gauge boson masses, the 3-point integrals (98) can immediately be reduced to 2-point integrals.

A similar comment applies to the 4-point functions. The general expressions are very lengthy and involved. We do not want to list them here but refer to the literature [20, 43]. For the situation of $e^+e^- \to f\bar{f}$ with light fermions they again become much simpler and will also be given in the corresponding places of the next chapter. In low energy processes the 4-point integrals shrink essentially to 2-point functions.

5 Standard Model one-loop expressions

As an application of the previous section we give here the explicit results for the fermion and vector boson self energies in the Standard Model and discuss their impact on the electroweak parameters. The self energies are of particular importance since they determine all the renormalization constants and hence provide the complete 1-loop renormalization of the Standard Model. The results are given in the $R_{\xi=1}$ gauge.

5.1 *Fermion self energies*

The diagrams contributing $\Sigma^f(k^2)$ for a given fermion species are displayed in Figure 1. f' denotes the isospin partner of the fermion f.

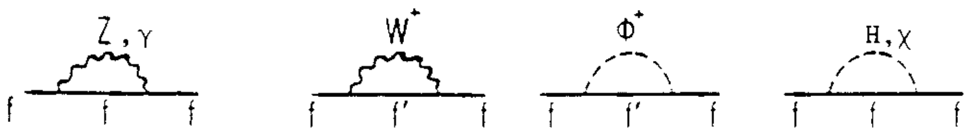


Fig. 1: *Fermion self energies*

We list the scalar coefficients $\Sigma^f_{V,A,S}$ in the decomposition of Σ^f in Eq. (51):

$$\begin{aligned}
\Sigma^f_V(k^2) &= -\frac{\alpha}{4\pi}\,[\,Q_f^2(2B_1(k^2,m_f,\lambda)+1) \\
&\quad +\frac{v_f^2+a_f^2}{4s_W^2c_W^2}(2B_1(k^2,m_f,M_Z)+1) \\
&\quad +\frac{1}{4s^2M_W^2}(m_f^2+m_{f'}^2)B_1(k^2,m_{f'},M_W)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{4s^2}(2B_1(k^2,m_{f'},M_W)+1) \\
& +\frac{m_f^2}{4s^2M_W^2}(B_1(k^2,m_f,M_H)+B_1(k^2,m_f,M_Z))\,] \\
\Sigma_A^f(k^2) \;=\; & -\frac{\alpha}{4\pi}[-\frac{2v_fa_f}{4s_W^2c_W^2}(2B_1(k^2,m_f,M_Z)+1) \\
& +\frac{1}{4s^2M_W^2}(m_{f'}^2-m_f^2)B_1(k^2,m_{f'},M_W) \\
& -\frac{1}{4s^2}(2B_1(k^2,m_{f'},M_W)-1)] \\
\Sigma_S^f(k^2) \;=\; & -\frac{\alpha}{4\pi}[Q_f^2(4B_0(k^2,m_f,\lambda)-2) \\
& +\frac{v_f^2-a_f^2}{4s_W^2c_W^2}(4B_0(k^2,m_f,M_Z)-2)+\frac{m_{f'}^2}{2s^2M_W^2}B_0(k^2,m_{f'},M_W) \\
& +\frac{m_f^2}{4s^2M_W^2}(B_0(k^2,m_f,M_Z)-B_0(k^2,m_f,M_H))] \qquad (108)
\end{aligned}
$$

Inserting these expressions into Eq.s (65-67) fixes the mass and field renormalization counter terms of the fermions.

In Eq. (108) the quantity λ has been introduced as a fictitious small photon mass in order to regularize the IR singularity from the diagram with photon exchange. All other diagrams are IR finite. For light fermions with m_f, $m_{f'} \ll M_{W,Z}$ one can neglect the Higgs contributions. For b quarks, only the diagrams with H and χ^0 can be neglected, but not the one where the charged Higgs component ϕ^+ goes together with the top quark.

5.2 *Vector boson self energies*

The diagrams contributing to the self energies of the photon, W, Z and the photon-Z transition are shown in Figure 2. We first consider the fermion loops.

Fermionic contributions:

Photon self energy:

We give the expression for a single fermion with charge Q_f and mass m. The total contribution is obtained by summing over all fermions. Evaluating the fermion loop diagram we obtain in the notation of section 4.2:

$$
\Sigma^{\gamma\gamma}(k^2) \;=\; \frac{\alpha}{\pi}Q_f^2\,\{-A(m)+\frac{k^2}{2}B_0(k^2,m,m)+2\,B_{22}(k^2,m,m)\}
$$

$$= \frac{\alpha}{3\pi} Q_f^2 \left\{ k^2 \left(\Delta - \log \frac{m^2}{\mu^2} \right) + (k^2 + 2m^2)\, \bar{B}_0(k^2, m, m) \;-\; \frac{q^2}{3} \right\}. \quad (109)$$

$\bar{B}_0$ denotes the finite function

$$\bar{B}_0(k^2, m, m) \;=\; -\int_0^1 dx \, \log \left(\frac{x^2k^2 - xk^2 + m^2}{m^2} - i\varepsilon \right) \qquad (110)$$

in the decomposition

$$B_0(k^2, m, m) \;=\; \Delta - \log \frac{m^2}{\mu^2} \;+\; \bar{B}_0(k^2, m, m)\,. \qquad (111)$$

The dimensionless quantity

$$\Pi^\gamma(k^2) \;=\; \frac{\Sigma^{\gamma\gamma}(k^2)}{k^2} \qquad (112)$$

is usually denoted as the photon "vacuum polarization". We list two simple expressions arising from Eq. (109) for special situations of practical interest:

- light fermions ($| k^2 | \gg m^2$):

$$\Pi^\gamma(k^2) = \frac{\alpha}{3\pi} Q_f^2 \left(\Delta - \log \frac{m^2}{\mu^2} + \frac{5}{3} - \log \frac{| k^2 |}{m^2} \;+\; i\pi\, \theta(k^2) \right) \qquad (113)$$

- heavy fermions ($| k^2 | \ll m^2$):

$$\Pi^\gamma(k^2) = \frac{\alpha}{3\pi} Q_f^2 \left(\Delta - \log \frac{m^2}{\mu^2} + \frac{k^2}{5m^2} \right) \qquad (114)$$

<u>Photon - Z mixing:</u>

Each charged fermion yields a contribution

$$\Sigma^{\gamma Z}(k^2) \;=\; -\frac{\alpha}{3\pi} \frac{v_f Q_f}{2 s_W c_W} \left\{ k^2 \left(\Delta - \log \frac{m^2}{\mu^2} \right) + (k^2 + 2m^2)\, \bar{B}_0(k^2, m, m) \;-\; \frac{q^2}{3} \right\}. \qquad (115)$$

As in the photon case, the fermion loop contribution vanishes for $k^2 = 0$.

<u>Z and W self energies:</u>

We give the formulae for a single doublet, leptons or quarks, with $m_\pm$, $Q_\pm$, $v_\pm$, $a_\pm$ denoting mass, charge, vector and axial vector coupling of the up(+) and the down(-) member. At the end, we have to perform the sum over the various doublets,

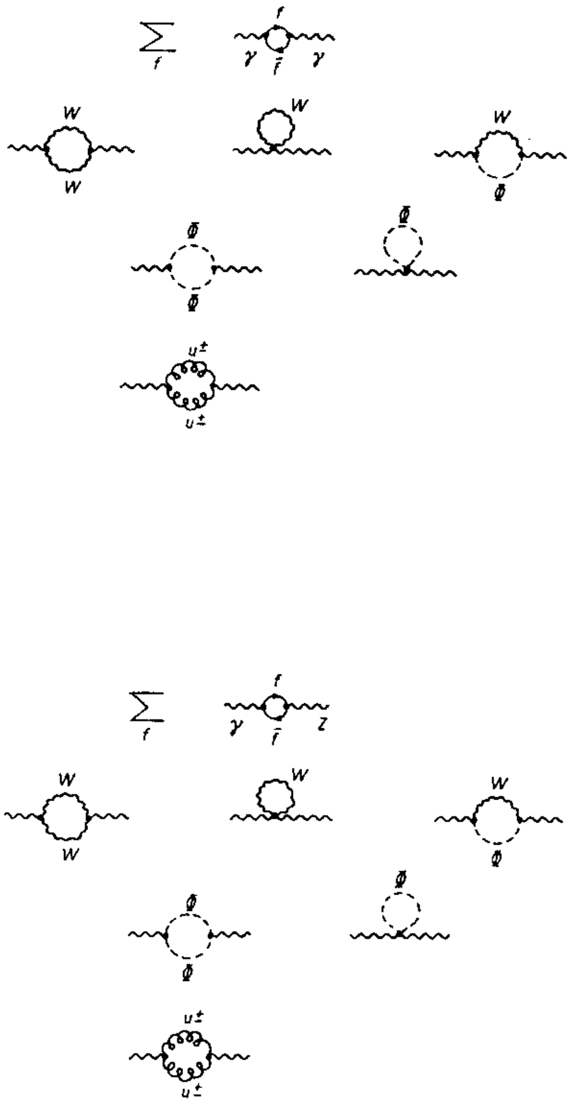


Fig. 2a: *Photon self energy and photon-Z transition*

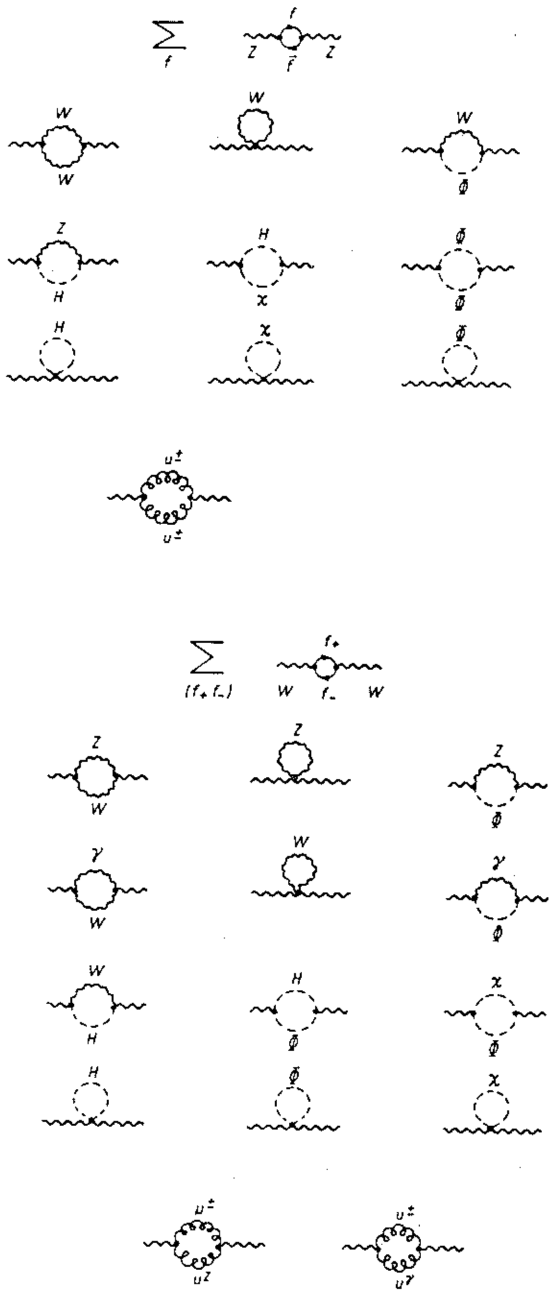


Fig. 2b: *W and Z self energies*

including color summation.

$$\Sigma^{ZZ}(k^2) = \frac{\alpha}{\pi} \sum_{f=+,-} \{ \frac{v_f^2 + a_f^2}{4s_W^2 c_W^2} \left[2B_{22}(k^2, m_f, m_f) + \frac{k^2}{2} B_0(k^2, m_f, m_f) - A(m_f) \right] - \frac{m_f^2}{8s_W^2 c_W^2} B_0(k^2, m_f^2, m_f^2) \}$$

$$\Sigma^{WW}(k^2) = \frac{\alpha}{\pi} \cdot \frac{1}{4s_W^2} \{ 2B_{22}(k^2, m_+, m_-) - \frac{A(m_+) + A(m_-)}{2} + \frac{k^2 - m_+^2 - m_-^2}{2} B_0(k^2, m_+, m_-) \} \tag{116}$$

Again, the following two cases are of particular practical interest:

- Light fermions:

In the light fermion limit $k^2 \gg m_\pm^2$ the$Z$ and W self-energies simplify considerably:

$$\Sigma^{ZZ}(k^2) = \frac{\alpha}{3\pi} \cdot \frac{v_+^2 + a_+^2 + v_-^2 + a_-^2}{4s_W^2 c_W^2} k^2 \left(\Delta - \log \frac{k^2}{\mu^2} + i\pi \right),$$

$$\Sigma^{WW}(k^2) = \frac{\alpha}{3\pi} \cdot \frac{k^2}{4s_W^2} \left(\Delta - \log \frac{k^2}{\mu^2} + i\pi \right). \tag{117}$$

- Heavy fermions:

Of special interest is the case of a heavy top quark which yields a large correction $\sim m_t^2$. In order to extract this part we keep for simplicity only those terms which are either singular or quadratic in the top mass $m_t \equiv m_+$ ($N_C = 3$):

$$\Sigma^{ZZ}(k^2) = N_C \frac{\alpha}{3\pi} \left\{ \frac{v_+^2 + a_+^2 + v_-^2 + a_-^2}{4s_W^2 c_W^2} k^2 - \frac{3m_t^2}{8s_W^2 c_W^2} \right\} \left(\Delta - \log \frac{m_t^2}{\mu^2} \right) + \cdots$$

$$\Sigma^{WW}(k^2) = N_C \frac{\alpha}{3\pi} \left\{ \frac{k^2}{4s_W^2} \left(\Delta - \log \frac{m_t^2}{\mu^2} \right) - \frac{3m_t^2}{8s_W^2} \left(\Delta - \log \frac{m_t^2}{\mu^2} + \frac{1}{2} \right) \right\} + \cdots \tag{118}$$

The quantity

$$\Delta\rho = \frac{\Sigma^{ZZ}(0)}{M_Z^2} - \frac{\Sigma^{WW}(0)}{M_W^2} \tag{119}$$

is finite as far as the heavy fermion contribution is considered which yields for the top quark:

$$\Delta\rho = N_C \frac{\alpha}{16\pi s_W^2 c_W^2} \frac{m_t^2}{M_Z^2}. \tag{120}$$

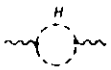
	$\Sigma^{\gamma\gamma}(k^2) = \frac{\alpha}{4\pi s_W^2}\cdot$	$\Sigma^{ZZ}(k^2) = \frac{\alpha}{4\pi s_W^2}\cdot$	$\Sigma^{\gamma Z}(k^2) = \frac{\alpha}{4\pi s_W^2}\cdot$	$\Sigma^{WW}(k^2) = \frac{\alpha}{4\pi s_W^2}\cdot$
	$A_1(k^2, M_W, M_W)\cdot s_W^2$	$c_W^2 A_1(k^2, M_W, M_W)$	$s_W c_W A_1(k^2, M_W, M_W)$	$c_W^2 A_1(k^2, M_W, M_W)$ $+s_W^2 A_1(k^2, 0, M_W)$
	$2s_W^2 A_2(M_W)$	$2c_W^2 A_2(M_W)$	$2c_W s_W A_2(M_W)$	$A_2(M_W) + c_W^2 A_2(M_Z)$
	$2s_W^2 M_W^2 B_0(k^2, M_W, M_W)$	$2s_W^4 M_Z^2 B_0(k^2, M_W, M_W)$	$-2\frac{s_W^3}{c_W} M_W^2 B_0(k^2, M_W, M_W)$	$s_W^4 M_Z^2 B_0(k^2, M_Z, M_W)$ $+s_W^2 M_W^2 B_0(k^2, 0, M_W)$
	0	$\frac{M_Z^2}{c_W^2} B_0(k^2, M_H, M_Z)$	0	$M_W^2 B_0(k^2, M_H, M_W)$
	0	$-\frac{1}{c_W^2} B_{22}(k^2, M_H, M_Z)$	0	$-B_{22}(k^2, M_H, M_W)$
	$-4s_W^2 B_{22}(k^2, M_W, M_W)$	$-\frac{(c_W^2 - s_W^2)^2}{c_W^2} B_{22}(k^2, M_W, M_W)$	$-2\frac{s_W}{c_W}(c_W^2 - s_W^2) B_{22}(k^2, M_W, M_W)$	$-B_{22}(k^2, M_Z, M_W)$
	$2s_W^2 B_{22}(k^2, M_W, M_W)$	$2c_W^2 B_{22}(k^2, M_W, M_W)$	$2s_W c_W B_{22}(k^2, M_W, M_W)$	$2c_W^2 B_{22}(k^2, M_Z, M_W)$ $+2s_W^2 B_{22}(k^2, 0, M_W)$
	$-2s_W^2 A(M_W)$	$-\frac{1}{4c_W^2}[A(M_H) + A(M_Z)]$ $-\frac{(c_W^2 - s_W^2)^2}{2c_W^2} A(M_W)$	$-\frac{s_W}{c_W}(c_W^2 - s_W^2) A(M_W)$	$-\frac{1}{4}[A(M_H) + A(M_Z)]$ $-\frac{1}{2} A(M_W)$

Table 1: Bosonic contributions to the vector boson self energies

Bosonic contributions:

The bosonic contributions to the vector boson self energies consist of the loop diagrams involving the gauge boson self interactions, the Higgs boson together with its unphysical components, and the Faddeev-Popov ghost fields. They are listed synoptically in Table 1 for the γ, Z, W self energies and the $\gamma - Z$ transition. The result for each part is the sum of the entries in the corresponding column, times the factor indicated in the first line. The functions A_1, A_2 appearing in table 1, are abbreviations for the following combinations of 1- and 2-point integrals:

$$\begin{aligned} A_1(k^2, m_1, m_2) &= A(m_1) + A(m_2) - (m_1^2 + m_2^2 + 4\,k^2)\,B_0(k^2, m_1, m_2) \\ &\quad -10\,B_{22}(k^2, m_1, m_2) \;+\; 2\,(m_1^2 + m_2^2 - \frac{k^2}{3})\,, \\ A_2(m) &= -3\,A(m) - 2\,m^2\,. \end{aligned}$$

5.3 *Electroweak parameter shifts*

We can now apply the results of 5.2 to discuss the contributions to the electroweak parameter shifts at the 1-loop level via the renormalization procedure. Such shifts are essentially the counter terms for the electric charge in Eq. (71) and for the electroweak mixing angle in Eq. (76). Since these counter terms are universal, they appear everywhere where in the lowest order expressions e resp. $\sin^2\theta_W$ is present. The shifts by the counter terms are not finite. However, their finite parts contain large terms which turn out to be the dominating contributions in the 1-loop corrections to the relations between the physical parameters.

$\Delta\alpha$ and effective charge:

The charge counter term in Eq. (71) contains the photon vacuum polarization at $k^2 = 0$. We split off the subtracted part evaluated at M_Z^2:

$$\begin{aligned} \Pi^\gamma(0) &= -\mathrm{Re}\Pi^\gamma(M_Z^2) + \Pi^\gamma(0) + \mathrm{Re}\,\Pi^\gamma(M_Z^2) \\ &= -\mathrm{Re}\,\hat{\Pi}^\gamma(M_Z^2) + \mathrm{Re}\,\Pi^\gamma(M_Z^2)\,. \end{aligned} \tag{121}$$

The subtracted quantity $\hat{\Pi}^\gamma(M_Z^2)$, which is the renormalized vacuum polarization according to Eq.s (112,47,64), is UV finite. Its fermionic content can be split into a leptonic and a hadronic part:

$$\mathrm{Re}\,\hat{\Pi}^\gamma_{ferm}(M_Z^2) = \mathrm{Re}\,\hat{\Pi}^\gamma_{lept}(M_Z^2) + \mathrm{Re}\,\hat{\Pi}^\gamma_{had}(M_Z^2)\,.$$

Heavy top quarks decouple from the subtracted vacuum polarization, as can be seen immediately from Eq. (114):

$$\hat{\Pi}^\gamma_{top}(M_Z^2) = \frac{\alpha}{\pi}\,Q_t^2\,\frac{M_Z^2}{5\,m_t^2}\,. \tag{122}$$

Whereas the leptonic content can easily be obtained from

$$\mathrm{Re}\,\hat{\Pi}^{\gamma}_{lept}(M_Z^2) = \sum_{l=e,\mu,\tau} \frac{\alpha}{3\pi}\left(\frac{5}{3} - \log\frac{M_Z^2}{m_l^2}\right), \tag{123}$$

no light quark masses are available as reasonable input parameters for the hadronic content. Instead, the 5 flavor contribution to $\hat{\Pi}^{\gamma}_{had}$ can be derived from experimental data with the help of a dispersion relation

$$\hat{\Pi}^{\gamma}_{had}(M_Z^2) = \frac{\alpha}{3\pi} M_Z^2 \int_{4m_\pi^2}^{\infty} ds' \frac{R^{\gamma}(s')}{s'(s' - M_Z^2 - i\varepsilon)} \tag{124}$$

with

$$R^{\gamma}(s) = \frac{\sigma(e^+e^- \to \gamma^* \to hadrons)}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

as an experimental quantity up to a scale s_1 and applying perturbative QCD for the tail region above s_1. Using e^+e^- data for the energy range below 40 GeV the integral (124) yields [44]

$$\mathrm{Re}\,\hat{\Pi}^{\gamma}_{had}(s_0) = -0.0282 \pm 0.0009 \tag{125}$$

for $s_0 = (92\,\mathrm{GeV})^2$. The error is almost completely due to the experimental data. Combining this result with the leptonic part one obtains

$$\mathrm{Re}\,\hat{\Pi}^{\gamma}_{ferm}(M_Z^2) = -0.0595 \pm 0.0009 \quad \text{for } M_Z = 91.187\,\mathrm{GeV}. \tag{126}$$

This finite quantity arising from the light fermion loops is independent of the structure of the electroweak model. It corresponds to a QED induced shift

$$\Delta\alpha = -\,\mathrm{Re}\,\hat{\Pi}^{\gamma}_{ferm}(M_Z^2) \tag{127}$$

in the electromagnetic fine structure constant:

$$\alpha \to \alpha(1 + \Delta\alpha)$$

which can be resummed according to the renormalization group accommodating all the leading logarithms of the type $\alpha^n \log^n(M_Z/m_f)$. The result is an effective fine structure constant at the Z mass scale:

$$\alpha(M_Z^2) = \frac{\alpha}{1 - \Delta\alpha} = \frac{1}{128.9 \pm 0.1}. \tag{128}$$

The ρ-parameter:

The ρ-parameter, which in the Standard Model is unity at the tree level, gets a deviation $\Delta\rho$ from 1 by radiative corrections. ρ has been defined as the ratio of the neutral to the charged current strength in neutrino scattering. It is modified by the quantity $\Delta\rho$ in Eq. (119) yielding the expression (120) for the contribution of the top quark. A general doublet of fermions with masses m_1, m_2 causes a shift of ρ by [45]

$$\Delta\rho_{ferm} = N_C \frac{\alpha}{16\pi s_W^2 c_W^2 M_Z^2} \left(m_1^2 + m_2^2 - \frac{2m_1^2 m_2^2}{m_1^2 - m_2^2} \log \frac{m_1^2}{m_2^2} \right). \qquad (129)$$

For the (t, b)-doublet, neglecting m_b, Eq. (120) is recovered.

It is important to note that this potentially large fermionic contribution to $\Delta\rho$ simultaneously constitutes the leading shift for the electroweak mixing angle according to Eq. (76) since there are no other terms $\sim m_f^2$ in the mass counter terms $\delta M_Z^2, \delta M_W^2$ besides those which are k^2-independent, hence leading to:

$$\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \simeq \frac{\Sigma^{ZZ}(0)}{M_Z^2} - \frac{\Sigma^{WW}(0)}{M_W^2} = \Delta\rho\,. \qquad (130)$$

There is also a Higgs contribution to $\Delta\rho$ which, however, is not UV finite by itself when derived from the diagrams involving the physical Higgs boson only. From table 1 one obtains:

$$\begin{aligned} \Delta\rho_H &= \frac{g_2^2}{16\pi^2} \cdot \frac{3}{4M_W^2} \left[M_W^2 (\Delta - \log \frac{M_W^2}{\mu^2}) - M_Z^2 (\Delta - \log \frac{M_Z^2}{\mu^2}) + \frac{5}{6}(M_Z^2 - M_W^2) \right. \\ &\left. + \frac{M_Z^2 M_H^2}{M_Z^2 - M_H^2} \log \frac{M_H^2}{M_Z^2} - \frac{M_W^2 M_H^2}{M_W^2 - M_H^2} \log \frac{M_H^2}{M_W^2} \right]. \end{aligned} \qquad (131)$$

From this expression the dependence on M_H for large Higgs masses $M_H \gg M_{W,Z}$ can be derived which, in contrast to heavy fermions, is only logarithmic [46]:

$$\Delta\rho_H \simeq \frac{g_2^2}{16\pi^2} \cdot \frac{3 s_W^2}{4 c_W^2} \log \frac{M_H^2}{M_W^2} + \cdots \qquad (132)$$

In the limit $s_W^2 \to 0, M_Z \to M_W$, where the $U(1)_Y$ is switched off, one finds $\Delta\rho_H = 0$. This is the consequence of the global $SU(2)_R$ symmetry of the Higgs Lagrangian ('custodial symmetry'), which is broken by the $U(1)_Y$ group. $\Delta\rho_H$ is thus a measure of the $SU(2)_R$ breaking by the weak hypercharge.

In contrast to the top term $\sim m_t^2$, the Higgs boson enters the shift of $\sin^2\theta_W$ not exclusively through $\Delta\rho$. There are additional M_H-dependent terms in $\delta M_Z^2, \delta M_W^2$ besides the ones in $\Sigma^{WW}(0)$ and $\Sigma^{ZZ}(0)$. The same remark holds for logarithmic top terms $\sim \log(m_t/M_W)$ which are not present in $\Delta\rho$.

The $M_W - M_Z$ interdependence:

Incorporating the parameter shifts

$$\alpha \to \alpha(1 + \Delta\alpha), \quad s_W^2 \to s_W^2 + c_W^2 \Delta\rho$$

with $\Delta\alpha, \Delta\rho$ from Eq.s (120), (127) into the relation (28), we obtain the approximate correlation at 1-loop

$$M_W^2 \left(1 - \frac{M_W^2}{M_Z^2}\right) = \frac{\pi\alpha}{\sqrt{2}G_\mu}\left(1 + \Delta\alpha - \frac{c_W^2}{s_W^2}\Delta\rho + \cdots\right) \tag{133}$$

between the vector boson masses and the other electroweak parameters α and G_μ, taking into account the large contributions from light and heavy fermions. The $\cdots$ indicate the residual terms belonging to the full calculation discussed in the next section.

The NC couplings:

In a similar way as done above, we obtain a universal shift in the overall normalization of the NC coupling constants in Eq.s (36,37)

$$\begin{aligned} \frac{e}{2s_W c_W} \quad \rightarrow \quad & \frac{e}{2s_W c_W}\left[1 + \frac{1}{2}\left(\Delta\alpha - \frac{c_W^2 - s_W^2}{s_W^2}\Delta\rho\right)\right] \\ & = (\sqrt{2}G_\mu M_Z^2)^{1/2}\left[1 + \frac{\Delta\rho}{2}\right] \\ & \rightarrow (\sqrt{2}G_\mu M_Z^2 \rho_f)^{1/2} \end{aligned} \tag{134}$$

The complete expressions for the normalization factor

$$\rho_f = 1 + \Delta\rho + \cdots$$

and the effective mixing angle

$$s_f^2 = s_W^2 + c_W^2 \Delta\rho + \cdots$$

in the Zff vertex between on-shell Z bosons and fermions will be be presented and discussed in the subsequent chapter on the Z resonance.

6 The muon lifetime and the gauge boson masses

6.1 *One-loop corrections to the muon lifetime*

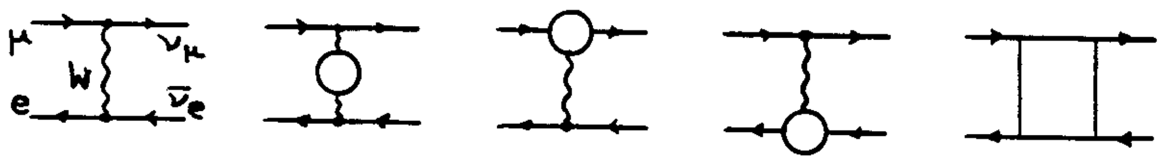
The interdependence between the gauge boson masses is established through the accurately measured muon lifetime or the Fermi coupling constant G_μ, respectively. Originally, the μ-lifetime τ_μ has been calculated within the framework of the effective 4-point Fermi interaction. If QED corrections are included one obtains the result [47]

$$\frac{1}{\tau_\mu} = \frac{G_\mu^2 m_\mu^5}{192\pi^3}\left(1 - \frac{8m_e^2}{m_\mu^2}\right)\left[1 + \frac{\alpha}{2\pi}(\frac{25}{4} - \pi^2)\right]. \tag{135}$$

The leading $2nd$ order correction is obtained by replacing

$$\alpha \to \alpha \left(1 + \frac{2\alpha}{3\pi} \log \frac{m_\mu}{m_e}\right) .$$

This formula is used as the defining equation for G_μ in terms of the experimental μ-lifetime. In lowest order, the Fermi constant is given by the Standard Model expression (26) for the decay amplitude. In 1-loop order, $G_\mu/\sqrt{2}$ is identified with the expression



$$\begin{aligned} \frac{G_\mu}{\sqrt{2}} &= \frac{e^2}{8s_W^2 M_W^2}\left[1 + \frac{\hat{\Sigma}^{WW}(0)}{M_W^2} + \delta_{VB}\right] \\ &\equiv \frac{e^2}{8s_W^2 M_W^2}\left[1 + \Delta r\right] . \end{aligned} \qquad (136)$$

The quantity $\Delta r(e, M_W, M_Z, M_H, m_f)$ is the UV and IR finite electroweak 1-loop correction to the muon decay amplitude in the Standard Model. Eq. (136) is the correlation between the vector boson masses and the other electroweak precision parameters α and G_μ. Due to the presence of m_t, M_H in Δr, this correlation becomes dependent on experimentally unknown quantities at the 1-loop level.

$\hat{\Sigma}^{WW}(0)$ is the renormalized W self energy from (47), evaluated at $k^2 = 0$, with the counter terms specified in (64). The term

$$\delta_{VB} = \frac{\alpha}{4\pi s_W^2}\left(6 + \frac{7 - 4s_W^2}{2s_W^2} \log c_W^2\right) \qquad (137)$$

summarizes the vertex corrections and box diagrams in the decay amplitude, more explicitly shown in Figure 3. A set of infra-red divergent "QED correction" graphs has been removed from this class of diagrams. These left-out diagrams, together with the real bremsstrahlung contributions, reproduce the QED correction factor of the Fermi model result in Eq. (135) and therefore have no influence on the relation between G_μ and the Standard Model parameters.

δ_{VB} has the structure

$$\delta_{VB} = \hat{F}^{We\nu} + \hat{F}^{W\mu\nu} - \frac{1}{2}\hat{\Pi}^{\nu_e} - \frac{1}{2}\hat{\Pi}^{\nu_\mu} + \delta_{box} \qquad (138)$$

with the following ingredients:

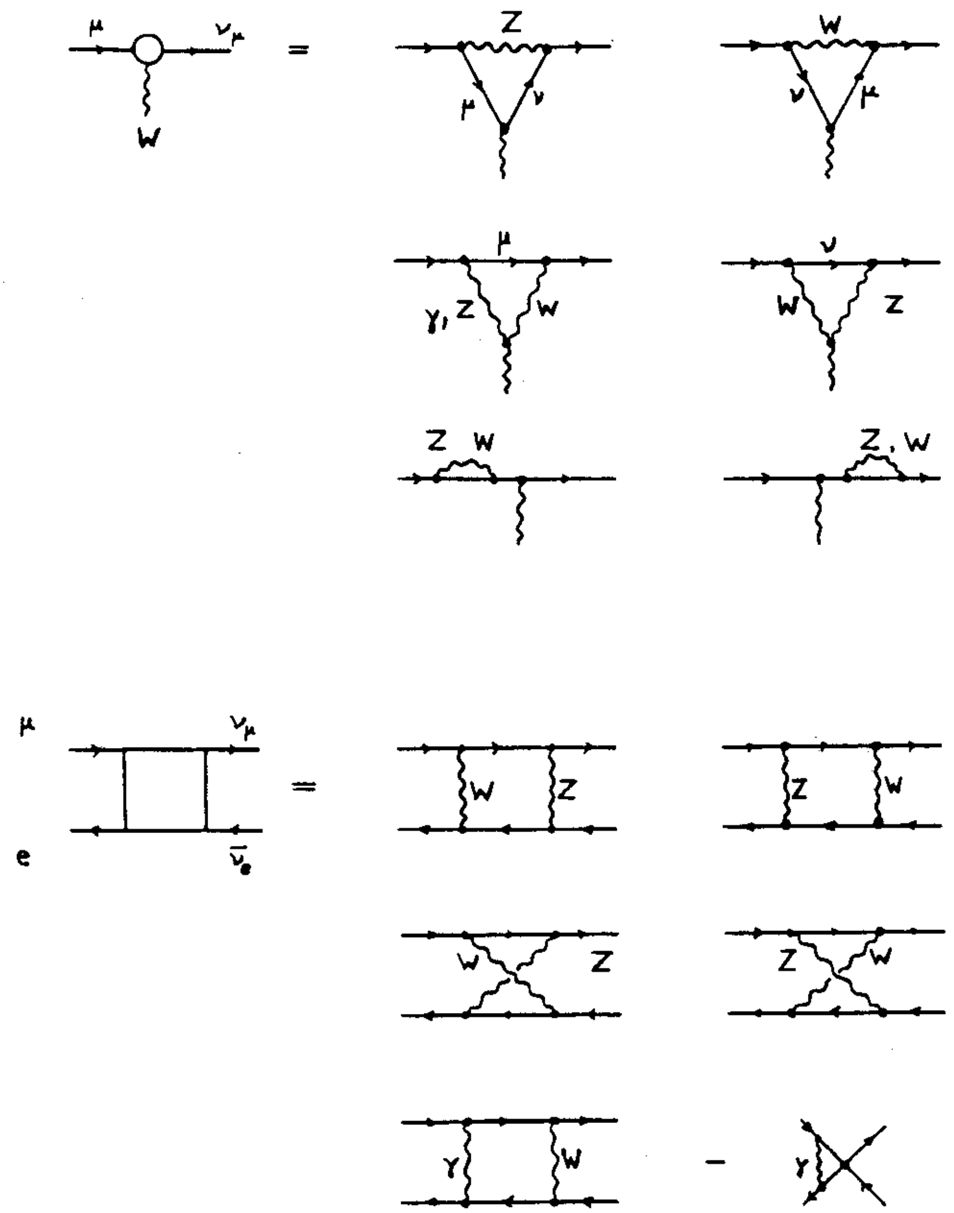


Fig. 3: *Vertex corrections with external self energies and box diagrams contributing to the 1-loop amplitude for $\mu \to \nu_\mu e \bar{\nu}_e$. For the $We\nu$-vertex the analogous sample of vertex corrections is present as well. Omitted are the "QED" diagrams with a photon in the external charged lepton lines and the photonic vertex correction to the Fermi amplitude is subtracted from the box diagram with photon exchange*

- the CC 1-loop form factors renormalized according to Eq. (59)

$$\hat{F}^{W\ell\nu} = F^{W\ell\nu}(0) + \delta Z_1^W - \delta Z_2^W + \delta Z_L^{(\ell)}, \quad \ell = e,\, \mu$$

  where $F^{W\ell\nu}(0)$ is the form factor in the sum of the CC vertex correction diagrams evaluated at $k^2 = 0$:

$$\Lambda_\mu^{W\ell\nu} = i\,\frac{e}{2\sqrt{2}s_W}\,\gamma_\mu(1-\gamma_5)\,F^{W\ell\nu}(0)\,.$$

  $\delta Z_L^{(\ell)}$ is the doublet field renormalization constant (66), evaluated from the e- and μ self energies without the virtual photon contribution (section 5.1).

- the finite wave function renormalization for the e and μ neutrino with

$$\hat{\Pi}^{\nu_\ell} = \Sigma_L^{\nu_\ell}(0) + \delta Z_L^{(\ell)}\,,$$

- the sum δ_{box} of the massive box diagrams and the γW-box with the IR subtraction as indicated in Figure 3.

Inserting the explicit expressions (64) into Eq. (136), the result for Δr gets the following representation:

$$\begin{aligned}\Delta r \;=\; & \Pi^\gamma(0) - \frac{c_W^2}{s_W^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) + \frac{\Sigma^{WW}(0) - \delta M_W^2}{M_W^2} \\ & +2\,\frac{c_W}{s_W}\,\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} + \frac{\alpha}{4\pi s_W^2}\left(6 + \frac{7-4s_W^2}{2s_W^2}\log c_W^2\right)\,. \qquad (139)\end{aligned}$$

All formulae required for calculating the quantities in Eq. (139) can be found in section 5.2.

By means of Eq.s (121) and (126) the fermionic part of the photon vacuum polarization $\Pi^\gamma(0)$ can be made explicit in Δr. There is also the contribution from a heavy top quark in terms of $\Delta\rho$ which enters through the relation (130) with $\Delta\rho$ given in Eq. (120). As can be seen from Eq. (139), there are no other large or potentially large terms in Δr besides those associated with $\Delta\alpha$ and $\Delta\rho$. When restricting to these two entries only, we recover the result already found in (133) by the discussion in section 5.3. Now we are able to write the following complete expression for Δr giving special emphasis on the dominant fermionic contributions:

$$\Delta r = \Delta\alpha - \frac{c_W^2}{s_W^2}\,\Delta\rho + (\Delta r)_{remainder}\,. \qquad (140)$$

$\Delta\alpha$, Eq. (127), contains the large logarithmic corrections from the light fermions and $\Delta\rho$ the leading quadratic correction from a large top mass. All other terms are collected in the $(\Delta r)_{remainder}$. It should be noted that the remainder also contains a term logarithmic in the top mass with a large coefficient:

$$(\Delta r)^{top}_{remainder} = -\frac{\alpha}{4\pi s_W^2}\left(\frac{c_W^2}{s_W^2} - \frac{1}{3}\right)\log\frac{m_t}{M_Z} + \cdots \tag{141}$$

Also the Higgs boson contribution is part of the remainder. For large M_H, it increases only logarithmically as it was already observed in the discussion of the ρ-parameter:

$$(\Delta r)^{Higgs}_{remainder} \simeq \frac{\alpha}{16\pi s_W^2}\cdot\frac{11}{3}\left(\log\frac{M_H^2}{M_W^2} - \frac{5}{6}\right). \tag{142}$$

The typical size of $(\Delta r)_{remainder}$ is of the order ~ 0.01.

6.2 *Higher order contributions*

For a top mass of 90 GeV the 1-loop quantity Δr is of the size 0.06 - 0.07 and we expect a 2-loop contribution typically of the order $(\Delta r)^2 \simeq 0.005$. This corresponds to a shift in the W mass of about 90 GeV which is the precision of the M_W measurement at LEP 200 and signals the need of going beyond the first order corrections in Eq. (136).

(i) Summation of large $\Delta\alpha$ terms:

The replacement of the $\Delta\alpha$-part

$$1 + \Delta\alpha \rightarrow \frac{1}{1-\Delta\alpha}$$

of the 1-loop result in Eq. (140) was already discussed in the context of the effective electromagnetic charge (128). It correctly takes into account all orders in the leading logarithmic corrections $(\Delta\alpha)^n$, as can be shown by renormalization group arguments [48] The evolution of the electromagnetic coupling with the scale μ is described by the renormalization group equation

$$\mu\frac{d\alpha}{d\mu} = -\frac{\beta_0}{2\pi}\alpha^2 \tag{143}$$

with the coefficient of the 1-loop β-function in QED

$$\beta_0 = -\frac{4}{3}\sum_{f\neq t} Q_f^2\,. \tag{144}$$

The solution of the RGE contains the leading logarithms in the resummed form as given in Eq. (128). It corresponds to a resummation of the iterated 1-loop vacuum polarization to all orders. The non-leading QED-terms of next order are numerically not significant. Thus, in a situation where large corrections are only due to the evolution of the electromagnetic charge between two very different scales set by m_f and M_Z, the resummed form

$$G_\mu = \frac{\pi\alpha}{\sqrt{2}M_W^2 s_W^2}\,\frac{1}{1-\Delta r} = \frac{\pi\alpha}{\sqrt{2}M_Z^2 c_W^2 s_W^2}\,\frac{1}{1-\Delta r} \tag{145}$$

with Δr in Eq. (140) represents a good approximation to the full result.

(ii) Summation of large $\Delta\rho$ terms:

In case of a heavy top, where also $\Delta\rho$ is large, the powers $(\Delta\rho)^n$ are not correctly resummed in Eq. (145). A result correct in the leading terms up to $O(\alpha^2)$ is instead given by [49] by the independent resummation

$$\frac{1}{1-\Delta r} \to \frac{1}{1-\Delta\alpha}\cdot\frac{1}{1+\frac{c_W^2}{s_W^2}\Delta\overline{\rho}} + (\Delta r)_{remainder} \tag{146}$$

where

$$\Delta\overline{\rho} = N_C\,\frac{G_\mu m_t^2}{8\pi^2\sqrt{2}}\cdot\left[1+\frac{G_\mu m_t^2}{8\pi^2\sqrt{2}}\,\rho^{(2)}\right] \tag{147}$$

incorporates the result from 2-loop 1-particle irreducible diagrams. For light Higgs bosons $M_H \ll m_t$, where M_H can be neglected, the coefficient

$$\rho^{(2)} = 19 - 2\pi^2 \tag{148}$$

was first calculated by Hoogeveen and van der Bij [50] and was recently confirmed by Barbieri et al. [51]. The general function $\rho^{(2)}$, valid for all Higgs masses, has been derived in [51]. For large Higgs masses $M_H > 2m_t$, a good approximation is given by the asymptotic expression with $r = (m_t/M_H)^2$ [51]

$$\begin{aligned}\rho^{(2)} \;=\;& \frac{49}{4} + \pi^2 + \frac{27}{2}\log r + \frac{3}{2}\log^2 r \\ & + \frac{r}{3}\left(2 - 12\pi^2 + 12\log r - 27\log^2 r\right) \\ & + \frac{r^2}{48}\left(1613 - 240\pi^2 - 1500\log r - 720\log^2 r\right)\,.\end{aligned} \tag{149}$$

Figure 4 shows the function $\rho^{(2)}$ together with the asymptotic formula. Except for very small Higgs masses, the results deviate significantly from the approximation for $M_H \to 0$.

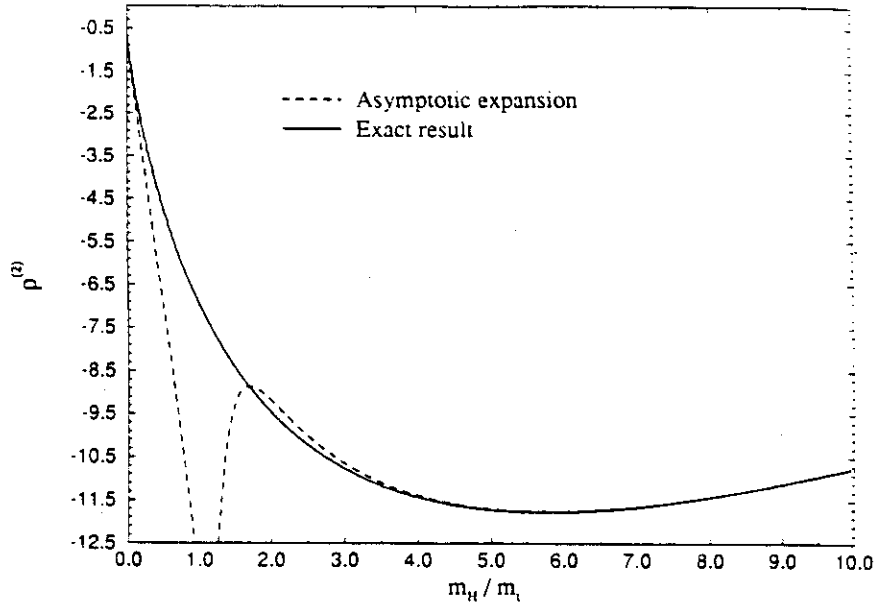


Fig. 4: *The function $\rho^{(2)}$ in Eq. (147), from [51].*

With the resummed ρ-parameter

$$\rho = \frac{1}{1-\Delta\overline{\rho}} \tag{150}$$

Eq. (146) is compatible with the following form of the $M_W - M_Z$ interdependence

$$G_\mu = \frac{\pi}{\sqrt{2}} \frac{\alpha(M_Z^2)}{M_W^2\left(1-\frac{M_W^2}{\rho M_Z^2}\right)} \cdot [1+(\Delta r)_{remainder}]\,. \tag{151}$$

It is interesting to compare this result with the corresponding lowest order $M_W - M_Z$ correlation in a more general model with a tree level ρ-parameter $\rho_0 \neq 1$: the tree-level ρ_0 enters in the same way as the ρ from a heavy top in the minimal model. The same applies for the quadratic mass terms from other particles like scalars or additional heavy fermions in isodoublets with large mass splittings. Hence, up to the small quantity $(\Delta r)_{remainder}$, they are indistinguishable from an experimental point of view ($\Delta\alpha$ is universal). In the minimal model, however, ρ is calculable in terms of m_t, M_H whereas ρ_0 is an *additional* free parameter.

(iii) QCD corrections for heavy top:

Virtual gluons contribute to the quark loops in the vector boson self-energies at the 2-loop level. For the light quarks this QCD correction is already contained in the result for the hadronic vacuum polarization from the dispersion integral, Eq. (124). Fermion loops involving the top quark get additional $O(\alpha\alpha_s)$ corrections which have been calculated perturbatively [52]. The dominating term for heavy top quarks is of the form $\alpha_s \alpha m_t^2$ and represents the QCD correction to the leading m_t^2 term of the ρ-parameter:

$$\Delta\rho \rightarrow \Delta\rho^{\alpha\alpha_s} = -\Delta\rho \cdot \frac{\alpha_s(m_t^2)}{\pi} \cdot \frac{2}{3}\left(\frac{\pi^2}{3}+1\right) . \tag{152}$$

This leading term already gives a sufficiently good approximation. This can be quantified in terms of a maximum deviation of M_W from the result based on the exact formulae which is less than 20 MeV. For heavy top masses the approximation becomes even better. As one of the 2-loop irreducible contributions to ρ, $\Delta\rho^{\alpha\alpha_s}$ has to be incorporated into $\Delta\overline{\rho}$ and resummed together with the electroweak 2-loop irreducible term as indicated in Eq. (151). Non-perturbative QCD effects in the gauge boson self energies associated with the $t\bar{t}$ threshold can be estimated with help of dispersive methods [53]. Expressed in terms of M_W, they shift the perturbative result by about +40 MeV for $m_t = 250$ GeV; for $m_t < 200$ GeV the influence on M_W is smaller than 25 GeV.

(iv) Non-leading higher order terms:

The modification of Eq. (146) by placing $(\Delta r)_{remainder}$ into the denominator

$$\frac{1}{1-\Delta r} \rightarrow \frac{1}{(1-\Delta\alpha)\cdot(1+\frac{c_W^2}{s_W^2}\Delta\overline{\rho}) \; - \; (\Delta r)_{remainder}} \tag{153}$$

correctly incorporates the non-leading higher order terms containing mass singularities of the type $\alpha^2 \log(M_Z/m_f)$ [54]

The treatment of the higher order reducible terms in Eq. (153) can be further refined by performing in $(\Delta r)_{remainder}$ the following substitution

$$\frac{\alpha}{s_W^2} \rightarrow \frac{\sqrt{2}}{\pi} G_\mu M_W^2 (1-\Delta\alpha) \tag{154}$$

in the expansion parameter of the combination

$$\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) - \Delta\rho$$

after cancellation of the UV singularity in the combination (139) or in the $\overline{MS}$ scheme with $\mu = M_Z$. This is discussed in [55] and is equivalent to the method described in [28] as well as to the recipe given at the end of ref. [49]. Numerically this modification is of some importance in the M_W-M_Z correlation for very heavy top quarks above 250 GeV. As an example, for $m_t = 300$ GeV one obtains a change in M_W by about 40 MeV.

A general comment, however, is in order: The refined treatment of the non-leading reducible higher order terms can be considered as an improvement only in case that the 2-loop irreducible non-leading terms are essentially smaller in size. Irreducible contributions of the type $\alpha G_\mu m_t^2 \log(m_t/M_Z)$ are unknown, and one has to rely on the assumption that the suppression by $1/N_C$ relative to the 2-loop reducible term is not compensated by a large coefficient. For bosonic 2-loop terms reducible and irreducible contributions are a priori of the same size and one does not gain from resumming 1-loop terms. In order to be on the safe side, the differences caused by the summation of non-leading reducible terms should be considered as a theoretical uncertainty at the level of 1-loop calculations improved by higher order leading terms.

6.3 *Numerical results and experimental data*

The correlation of the electroweak parameters, complete at the one-loop level and with the proper incorporation of the leading higher order effects, is given by the following equation:

$$\begin{aligned} M_W^2\left(1-\frac{M_W^2}{M_Z^2}\right) &= \frac{\pi\alpha}{\sqrt{2}G_\mu}\cdot\frac{1}{(1-\Delta\alpha)\cdot(1+\frac{c_W^2}{s_W^2}\Delta\overline{\rho}) - (\Delta r)_{remainder}} \\ &\equiv \frac{\pi\alpha}{\sqrt{2}G_\mu}\cdot\frac{1}{1-\Delta r}. \end{aligned} \tag{155}$$

The Δr in Eq. (155) is an effective quantity beyond the 1-loop order, introduced to obtain the formal analogy to the naively resummed first order result in Eq. (145). $\Delta\overline{\rho}$ includes the 2-loop irreducible electroweak and QCD corrections to the ρ-parameter:

$$\begin{aligned} \Delta\overline{\rho} &= \Delta\rho^{(1)} + \Delta\rho^{(2)} \\ &= N_C\,\frac{G_\mu m_t^2}{8\pi^2\sqrt{2}}\cdot\left[1+\frac{G_\mu m_t^2}{8\pi^2\sqrt{2}}\,\rho^{(2)} - \frac{\alpha_s(m_t^2)}{\pi}\cdot\frac{2}{3}\left(\frac{\pi^2}{3}+1\right)\right]. \end{aligned} \tag{156}$$

The correlation (155) allows us to predict a value for the W mass after the other parameters have been specified. These predicted values for M_W are put together in table 2 for various Higgs and top masses. The present experimental value for the W mass from the combined UA2 and CDF results [2] is

$$M_W^{exp} = 80.14 \pm 0.26\,\text{GeV}. \tag{157}$$

m_t	$M_H = 60$	100	300	1000
90	79.952	79.925	79.854	79.760
120	80.109	80.082	80.010	79.915
150	80.275	80.248	80.173	80.078
180	80.462	80.433	80.355	80.257
210	80.674	80.643	80.557	80.454
240	80.912	80.879	80.783	80.671

Table 2: The W mass M_W as predicted by the Standard Model for $M_Z = 91.187$ GeV and various top and Higgs masses, based on Eq. (155). The refinement described in Eq. (154) was taken into account. Nonperturbative QCD effects associated with the $t\bar{t}$ threshold have been neglected. All masses are in GeV

We can define the quantity Δr also as a physical observable by

$$\Delta r = 1 - \frac{\pi\alpha}{\sqrt{2}G_\mu} \frac{1}{M_W^2\left(1 - \frac{M_W^2}{M_Z^2}\right)}. \tag{158}$$

Experimentally, it is determined by M_Z and the ratio M_W/M_Z. Theoretically, it can be computed from M_Z, G_μ, α after specifying the masses M_H, m_t by solving Eq. (155). In Figure 5 we display the prediction for Δr as a function of m_t in various steps: the first order calculation based on Eq. (145) with the lowest order Δr, then including the electroweak higher order terms on the basis of Eq. (146), and finally including also the QCD corrections related to m_t. Both electroweak and QCD higher order effects yield a positive shift to Δr and thus diminish the slope of the first order dependence on m_t for large top masses. The effect on Δr coming from the modified $\rho^{(2)}$ in Eq. (147) for large M_H is shown in Figure 6. It causes an additional weakening of the sensitivity to m_t for large Higgs masses.

The theoretical prediction for Δr for various Higgs and top masses is displayed in Figure 7. For comparison with data, the experimental 1σ limits from the direct measurements of M_Z at LEP and M_W/M_Z in $p\bar{p}$ are indicated. The experimental input from LEP [1, 4] and from the combined UA2 and CDF results [2] is

$$M_Z = 91.187 \pm 0.007\text{GeV}, \;\; s_W^2 = 0.2275 \pm 0.0052\,.$$

For $M_H < 1$ TeV these results constrain the top mass to the range $m_t < 203$ GeV at the 1σ level. The present experimental error does not allow a sensitivity to the Higgs mass. Precision measurements of M_W at LEP 200 will pin down the error to $\delta\Delta r = 0.006$ (0.004 with high luminosity). This would determine m_t with an accuracy of about $\delta m_t = 10$ GeV. A still inherent uncertainty from the unknown Higgs mass (with $M_H > 60$ GeV), however, would give an an additional theoretical error of ± 17 GeV. The expected precision in the determination of Δr matches the

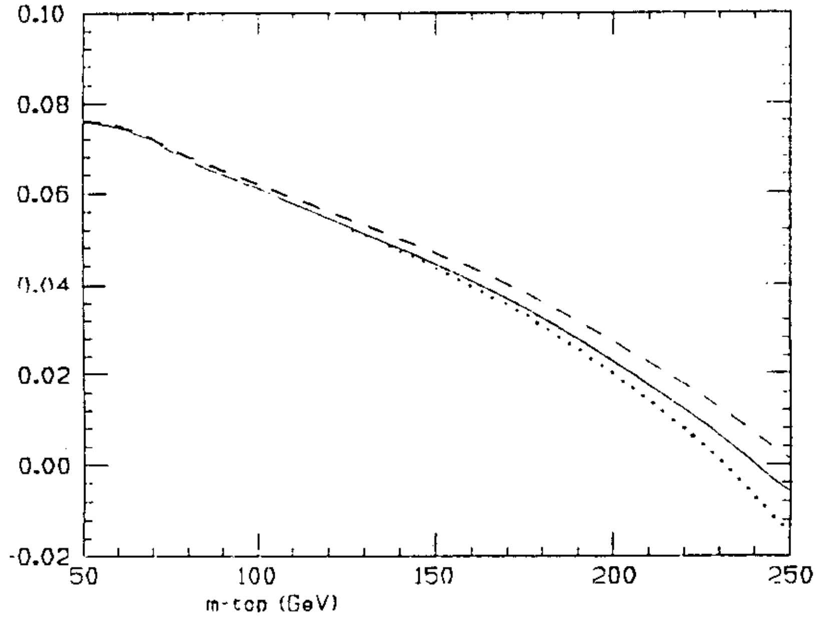


Fig. 5: *Δr in $O(\alpha)$ (dotted), in $O(\alpha^2)$ (full), and in $O(\alpha^2 + \alpha\alpha_s)$ (dashed).*
$M_Z = 91.187$ GeV, $M_H = 300$ GeV.

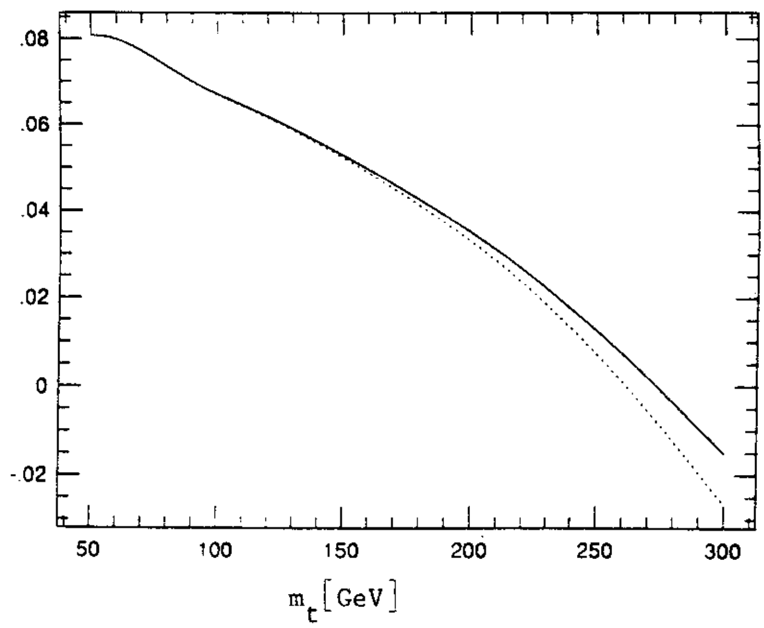


Fig. 6: *Δr in $O(\alpha^2 + \alpha\alpha_s)$ for $M_H = 1$ TeV with the Higgs dependent ρ-parameter (full) and the approximation (148) (dashed). $M_Z = 91.187$ GeV*

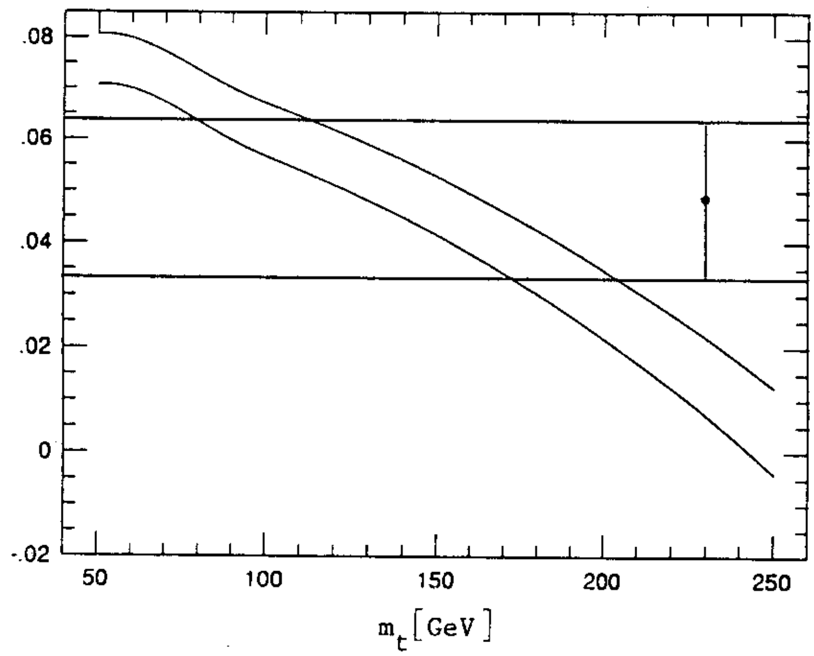


Fig. 7: Δr *as a function of the top mass for* $M_H = 60, 1000$ *GeV (lower. upper line).* $M_Z = 91.187 \pm .007$ *GeV.* 1σ *bounds with* $s_W^2 = 0.2275 \pm 0.0052$ *from combined UA2 and CDF results [2].*

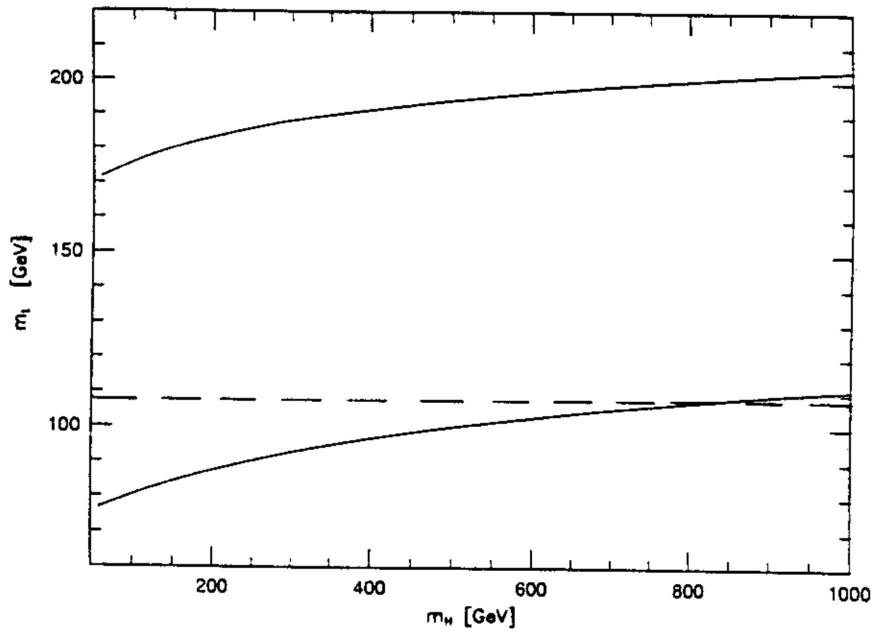


Fig. 8: *Sensitivity of the top mass bounds from* Δr *on the Higgs mass. The allowed* m_t *range is between the curves. The bound on* m_t *from the direct search is also indicated.*

size of $(\Delta r)_{remainder}$ and thus will provide some sensitivity also to the Higgs mass in case that the top quark would be discovered experimentally. For virtual Higgs effects, however, the observables from the Z resonance are more suitable.

The bounds on m_t, following from the experimental constraint

$$(\Delta r)_{exp} = 0.0489 \pm 0.0153$$

depend on the Higgs mass. This dependence is illustrated in Figure 8 . The allowed m_t range is further restricted by the bound [10, 11] from the direct search.

7 Renormalization schemes

In a Quantum Field Theory like the electroweak standard model the starting point for perturbative calculations is the Lagrangian with a set of free mass and coupling parameters. The general discussion of renormalization in Quantum Field Theory has shown that the freedom in parametrizing the theory can be used to introduce convenient renormalization constants, or counter terms, equivalently, and to absorb the divergences in the calculation of S-matrix elements or Green functions. It is also possible to deal with the bare parameters of the theory for relating measureable quantities to each other, but the bare parameters are cutoff dependent and hence have no simple physical interpretation.

A renormalization scheme is a choice of definite procedures for dealing with the parameters of the theory, together with the infinities from the loop amplitudes, in terms of measureable physical quantities. In a more general sense, it comprises the choice of the regularization procedure, the way of treating field renormalization, the specification of the gauge fixing terms and the respective FP ghost part, and a set of prescriptions how the formal parameters can be measured.

Before one can make predictions from the theory, a set of independent parameters has to be determined from experiment. This can either be done for the bare quantities or for renormalized parameters which have a simple physical interpretation. In a more restrictive sense, a renormalization scheme characterizes a specific choice of experimental data points to be used as input defining the basic parameters of the Lagrangian in terms of which the perturbative calculation of physical amplitudes is performed.

Predictions for the relations between physical quantities do not depend on the choice of a specific renormalization scheme if we perform the calculation to all orders in the perturbative expansion. Practical calculations, however, are obtained from truncated perturbation series, making the predictions depend on the chosen set of basic parameters and thus leading to a scheme dependence.

Differences between various schemes are formally of higher order than the one under consideration. To make this obvious, we consider a simplified model with only a single coupling paramater α. Calculation of a 1-loop amplitude for a process with the lowest order amplitude $M^{(0)} = \alpha^2 A_0$ yields

$$M^{(1)} = \alpha^2 A_0 [1 + b\alpha].$$

In another scheme with α' different from α by a 1-loop term

$$\alpha' = \alpha [1 + a\alpha]$$

the result is

$$M'^{(1)} = \alpha'^2 A_0 [1 + b'\alpha'].$$

After insertion of α', with $b' = b - 2a$, one gets

$$M'^{(1)} = M^{(1)} + \alpha^2 A_0 [3b'a\,\alpha^2 + O(\alpha^3)].$$

Without an explicit calculation of the $O(\alpha^2)$ correction the difference $M'^{(1)} - M^{(1)}$ has to be considered as an uncertainty. The study of the scheme dependence of the perturbative results, after improvement by resumming the leading terms, allows us to estimate the missing higher order contributions.

Parametrizations or 'renormalization schemes' frequently used in electroweak calculations are:

1. the on-shell (OS) scheme with

$$\alpha,\ M_W,\ M_Z,\ m_f,\ M_H$$

2. the G_μ scheme with the basic parameters

$$\alpha,\ G_\mu,\ M_Z,\ m_f,\ M_H$$

3. the low energy scheme with the mixing angle as a basic parameter defined in neutrino-electron scattering:

$$\alpha,\ G_\mu,\ \sin^2\theta_{\nu e},\ m_f,\ M_H$$

4. the $*$ scheme where the bare parameters e_0, G_μ^0, s_0^2 are eliminated and replaced in terms of dressed running (k^2-dependent) parameters

$$e_*^2(k^2),\ G_{\mu *}(k^2),\ s_*^2(k^2);\ m_f,\ M_H$$

5. the $\overline{MS}$-scheme.

Some details on the $\overline{MS}$ scheme will be given in the next subsection, followed by a brief discussion of the other renormalization schemes.

7.1 *The $\overline{MS}$-scheme*

The modified minimal subtraction scheme ($\overline{MS}$-scheme) [32, 33, 34, 35] is one of the simplest ways to obtain finite 1-loop expressions by performing the substitution

$$\frac{2}{\epsilon} - \gamma + \log 4\pi + \log \mu^2 \;\rightarrow\; \log \mu^2_{\overline{MS}}$$

in the divergent parts of the loop integrals, Eq. (95). Formally, the $\overline{MS}$ self energies and vertex corrections are obtained by splitting the bare masses and couplings into $\overline{MS}$ parameters and counter terms

$$M_0^2 = \hat{M}^2 + \delta\hat{M}^2, \quad e_0 = \hat{e} + \delta\hat{e}, \tag{159}$$

where the counter terms together with field renormalization constants

$$1 + \delta\hat{Z}_i$$

are defined in such a way that they absorb the singular parts proportional to

$$\Delta = \frac{2}{\epsilon} - \gamma + \log 4\pi\,.$$

As a consequence, self energies and vertex corrections in the $\overline{MS}$-scheme depend on the arbitrary scale μ.

Perturbative calculations start from the Lagrangian with the formal $\overline{MS}$ parameters

$$\mathcal{L}(\hat{e}, \hat{M}_W, \hat{M}_Z, \ldots).$$

The $\overline{MS}$ parameters fulfill the same relations as the corresponding bare parameters. In particular, the mixing angle in the $\overline{MS}$-scheme, denoted by $\hat{s}^2$, can be expressed in terms of the $\overline{MS}$ masses of W and Z in the following way:

$$\hat{s}^2 = 1 - \frac{\hat{M}_W^2}{\hat{M}_Z^2}\,. \tag{160}$$

The relation of the $\overline{MS}$ parameters to the conventional OS-parameters is obtained by calculating the dressed vector boson propagators and the dressed electron-photon vertex in the Thomson limit in the $\overline{MS}$-scheme and identifying the poles with the OS masses and the electromagnetic coupling with the classical charge.

• The $\overline{MS}$ charge:

The $\overline{MS}$ analogon of the OS charge renormalization condition Eq. (71) reads:

$$\hat{e}\left[1-\frac{1}{2}\,\Pi^{\gamma}_{\overline{MS}}(0)+\frac{\hat{s}}{\hat{c}}\,\frac{\Sigma^{\gamma Z}_{\overline{MS}}(0)}{\hat{M}_Z^2}\right] = e. \tag{161}$$

The l.h.s. is the coupling constant of the dressed electromagnetic vertex in the Thomson limit which has to be identified with the classical charge.

The $\overline{MS}$ self energies in Eq. (161) read explicititly:

$$\begin{aligned}
\Pi^{\gamma}_{\overline{MS}}(0) &= \frac{\hat{e}^2}{16\pi^2}\,A^{\gamma}(0), \\
A^{\gamma}(0) &= \frac{4}{3}\sum_f Q_f^2\,\log\frac{\mu^2}{m_f^2}+3\,\log\frac{M_W^2}{\mu^2}-\frac{2}{3}, \\
\frac{\hat{s}}{\hat{c}}\,\frac{\Sigma^{\gamma Z}_{\overline{MS}}(0)}{\hat{M}_Z^2} &= -\frac{\hat{e}^2}{8\pi^2}\,\log\frac{M_W^2}{\mu^2}.
\end{aligned} \tag{162}$$

A natural scale for electroweak physics is given by $\mu = M_Z$. Hence, the correlation between e and $\hat{e}$ involves large logarithms from the light fermions which can be resummed according to the RGE (143). The bosonic terms are small. Resummation leads to the relation

$$e^2 = \frac{\hat{e}^2}{1+\frac{\hat{e}^2}{16\pi^2}\left[A^{\gamma}(0)+4\,\log\frac{M_W^2}{\mu^2}\right]}. \tag{163}$$

Inverting this equation yields the $\overline{MS}$ charge expressed in terms of the OS charge

$$\hat{e}^2 = \frac{e^2}{1-\frac{e^2}{16\pi^2}\left[A^{\gamma}(0)+4\,\log\frac{M_W^2}{\mu^2}\right]}. \tag{164}$$

Choosing $\mu = M_Z$ we can evaluate the expression in (165) to obtain the $\overline{MS}$ fine structure constant at the Z mass scale

$$\hat{\alpha} = \frac{\alpha}{1-\Delta\hat{\alpha}} \tag{165}$$

with the value

$$\Delta\hat{\alpha} = 0.0684 \pm 0.0009 - \frac{8\alpha}{9\pi}\log\frac{m_t}{M_Z}+\frac{\alpha}{2\pi}\left(\frac{7}{2}\log c_W^2-\frac{1}{3}\right). \tag{166}$$

The first term is due to the light fermions. It can be obtained from the quantity in Eq. (127) by adding the constant term

$$\frac{\alpha}{\pi}\left(\frac{5}{3}+\frac{55}{27}(1+\frac{\alpha_s}{\pi})\right).$$

The uncertainty in Eq. (166) is the hadronic uncertainty of $\Delta\alpha$ in Eq. (127).

$\hat{\alpha}$ has to be distinguished from the effective charge at the Z scale introduced in Eq. (128) which contains only the light fermion contributions. A heavy top quark decouples in $\Delta\alpha$ according to Eq. (122), but does not decouple in $\Delta\hat{\alpha}$. Numerically one finds

$$\begin{aligned} \alpha(M_Z^2)^{-1} &= 128.8 \pm 0.1 \\ (\hat{\alpha})^{-1} &= 127.8 \; - \; 128.0 \pm 0.1 \end{aligned} \tag{167}$$

The variation in $\hat{\alpha}$ in Eq. (167) corresponds to a top mass range from $m_t = 90$ GeV to 250 GeV.

- The $\overline{MS}$ masses:

The $\overline{MS}$ mass parameters $\hat{M}_W^2, \hat{M}_Z^2$ enter the corresponding transverse propagators together with the self energies as follows ($V = W, Z$):

$$D_V = \frac{1}{k^2 - \hat{M}_V^2 + \Sigma_{\overline{MS}}^{VV}(k^2)} \tag{168}$$

The OS-masses fulfill the pole conditions

$$M_V^2 - \hat{M}_V^2 + \mathrm{Re}\,\Sigma_{\overline{MS}}^{VV}(M_V^2) = 0 \tag{169}$$

yielding $\hat{M}_V^2$ expressed in terms of the OS-masses:

$$\hat{M}_V^2 = M_V^2 + \mathrm{Re}\,\Sigma_{\overline{MS}}^{VV}(M_V^2). \tag{170}$$

The mass parameters $\hat{M}_V^2$ are μ-dependent. We can choose $\mu = M_Z$ as the natural scale for electroweak calaculations, as done also for $\hat{\alpha}$.

The self energies $\Sigma_{\overline{MS}}$ are obtained from the expressions given in section 5.2 by dropping everywhere the singular term Δ and substituting

$$e \to \hat{e}, \;\; s_W \to \hat{s}, \;\; c_W \to \hat{c}$$

in the couplings, with $\hat{c}^2 = 1 - \hat{s}^2$. It is convenient to remove the overall normalization factors and to write for the real parts:

$$\begin{aligned} \mathrm{Re}\,\Sigma_{\overline{MS}}^{WW} &= \frac{\hat{e}^2}{\hat{s}^2} A_W(k^2), \\ \mathrm{Re}\,\Sigma_{\overline{MS}}^{ZZ} &= \frac{\hat{e}^2}{\hat{s}^2\hat{c}^2} A_Z(k^2). \end{aligned} \tag{171}$$

- The $\overline{MS}$ mixing angle:

The mixing angle $\hat{s}^2$ in the $\overline{MS}$-scheme, defined in Eq. (160), can be related to the OS mixing angle $s_W^2 = 1 - M_W^2/M_Z^2$ by substituting $\hat{M}_{W,Z}^2$ from Eq. (170), yielding

$$\hat{s}^2 = s_W^2 + c_W^2\, X_{\overline{MS}}, \quad \hat{c}^2 = c_W^2\,(1 - X_{\overline{MS}}) \tag{172}$$

with

$$X_{\overline{MS}} = \frac{\hat{e}^2}{\hat{s}^2}\left(\frac{A_W(M_W^2)}{M_W^2} - \frac{A_Z(M_Z^2)}{\hat{c}^2 M_Z^2}\right)\left(1 - \frac{\hat{e}^2}{\hat{s}^2}\,\frac{A_Z(M_Z^2)}{\hat{c}^2 M_Z^2}\right)^{-1}. \tag{173}$$

Making use of the property

$$X_{\overline{MS}} = \frac{\hat{e}^2}{\hat{s}^2}\,\frac{A_W(M_W^2)}{M_W^2} - (1 - X_{\overline{MS}})\,\frac{\hat{e}^2}{\hat{s}^2}\,\frac{A_Z(M_Z^2)}{\hat{c}^2 M_Z^2}$$

the relation (172) can simplified:

$$\hat{s}^2 = s_W^2 + \frac{\hat{e}^2}{\hat{s}^2}\,\frac{A_Z(M_Z^2) - A_W(M_W^2)}{M_Z^2}. \tag{174}$$

The leading 2-loop irreducible contributions are incorporated by adding in (174) the extra term $c_W^2\,\Delta\rho^{(2)}$ with $\Delta\rho^{(2)}$ from Eq. (156).

Eq. (174) determines $\hat{s}^2$ in terms of the OS parameters. $\hat{e}^2$ has to be taken from Eq. (164) or (165), respectively, for $\mu = M_Z$. Numerical values for $\hat{s}^2$ (with $\mu = M_Z$) are listed in table 3 together with the corresponding values for the OS counter part s_W^2.

One can obtain $\hat{s}^2$ also in a more direct way from the experimental data points α, G_μ, M_Z, without passing first through the OS-calculation, by deriving the effective Fermi constant in the $\overline{MS}$-scheme

$$\frac{G_\mu}{\sqrt{2}} = \frac{\hat{e}^2}{8\,\hat{s}^2\hat{c}^2\hat{\rho}\,M_Z^2}\cdot\frac{1}{1 - \Delta\hat{r}} \tag{175}$$

where

$$\begin{aligned}
\Delta\hat{r} &= \frac{\hat{e}^2}{\hat{s}^2}\,\frac{A_W(0) - A_W(M_W^2)}{M_W^2} + \hat{\delta}_{VB}, \\
\hat{\delta}_{VB} &= \frac{\hat{\alpha}}{4\pi\hat{s}^2}\left[6 + \frac{7 - 5s_W^2 + \hat{s}^2(3c_W^2/\hat{c}^2 - 10)}{2s_W^2}\,\log c_W^2\right],
\end{aligned} \tag{176}$$

together with

$$\begin{aligned}
M_W^2 &= \hat{c}^2\hat{\rho}M_Z^2, \\
\hat{\rho} &= \frac{1}{1 - X_{\overline{MS}}}.
\end{aligned} \tag{177}$$

m_t (GeV)	M_H (GeV)	s_W^2	$\hat{s}^2$
90	60	0.2312	0.2335
90	300	0.2331	0.2343
90	1000	0.2349	0.2350
120	60	0.2282	0.2329
120	300	0.2301	0.2338
120	1000	0.2319	0.2345
150	60	0.2250	0.2322
150	300	0.2270	0.2330
150	1000	0.2288	0.2337
180	60	0.2214	0.2312
180	300	0.2235	0.2321
180	1000	0.2254	0.2328
210	60	0.2173	0.2301
210	300	0.2196	0.2311
210	1000	0.2215	0.2318
240	60	0.2127	0.2289
240	300	0.2152	0.2299
240	1000	0.2173	0.2307

Table 3: The mixing angles s_W^2 and $\hat{s}^2$ in the on-shell and in the $\overline{MS}$-scheme for $M_Z = 91.187$ GeV and various top and Higgs masses.

For given parameters $\alpha, G_\mu, M_Z, m_t, M_H$ the solution of this set of equations yields the quantities $\hat{s}^2, \hat{\rho}$ together with M_W. $\Delta\hat{r}$ is a small correction and has only a mild dependence on the top and Higgs masses. For the m_t, M_H range allowed in Figure 8 one has

$$\Delta\hat{r} = 0.0050 \pm 0.0034 \tag{178}$$

where the variation is due to the unknown mass parameters.

The term $\hat{\delta}_{VB}$ in $\Delta\hat{r}$ is the vertex and box correction to the muon decay amplitude in the $\overline{MS}$-scheme [34]. The given expression refers to a mixed $\overline{MS}$ - on-shell calculation of the loop diagrams where $\overline{MS}$-couplings are used but on-shell masses in the propagators. Numerically the differences to the corresponding expression exclusively with $\overline{MS}$ parameters is insignificant ($< 3 \cdot 10^{-4}$). The main difference to the on-shell quantity δ_{VB} in Eq. (137) (besides the parametrization) is the extra additive term

$$-\frac{\hat{\alpha}}{\pi}\log c_W^2 \equiv -\frac{\hat{\alpha}}{\pi}\log\frac{M_W^2}{\mu^2} \quad \text{for } \mu = M_Z$$

arising from the UV singularity in the sum of the diagrams.

The Standard Model prediction for $\hat{s}^2$ following from the mass range $m_t >$ 90 GeV and 60 GeV $< M_H <$ 1000 GeV together with the constraint from the experimental values for M_Z, M_W is given by

$$\hat{s}^2 = 0.2330 \pm 0.0016 . \tag{179}$$

This includes the uncertainty induced by the hadronic vacuum polarization in Eq. (166) resp. (126).

The $\overline{MS}$ quantities $\hat{\alpha}$, $\hat{s}^2$ are formal parameters which have no simple relation to physical quantities. The interest in these parameters is based on two important features:

- They are universal, i.e. process independent, and take into account the universal large effects from fermion loops. Expressing the NC coupling constants for the Zff vertices in terms of $\hat{\alpha}$, $\hat{s}^2$ yields a good approximation to the complete results (134):

$$\begin{aligned} \sqrt{2}G_\mu M_Z^2 \rho_f &= \frac{\hat{e}^2}{4\,\hat{s}^2\hat{c}^2}\,(1+\delta\hat{\rho}_f), \\ s_f^2 &= \hat{s}^2 + \delta\hat{s}_f^2 . \end{aligned} \tag{180}$$

  The flavor dependent residual corrections $\delta\hat{\rho}_f$ and $\delta\hat{s}_f^2$ are small and practically independent of m_t and M_H. An exception is the Zbb vertex, where also non-universal large top terms are present [56].

- The knowledge of the values for $\hat{\alpha}$ and $\hat{s}^2$ at the Z scale allows the extrapolation of the SU(2) and U(1) couplings

$$\hat{\alpha}_1(\mu^2) = \frac{\hat{\alpha}(\mu^2)}{\hat{c}^2(\mu^2)}, \quad \hat{\alpha}_2(\mu^2) = \frac{\hat{\alpha}(\mu^2)}{\hat{s}^2(\mu^2)} \tag{181}$$

  to large mass scales and, together with the strong coupling constant $\alpha_s(\mu^2)$ in the $\overline{MS}$-scheme, to test scenarios of Grand Unification. In particular the minimal SU(5) model of Grand Unfication predicts with α and α_s as input [57]:

$$\hat{s}^2_{SU(5)}(M_Z^2) = 0.2102^{+0.0037}_{-0.0031}$$

  which is in disagreement with the result (179). Supersymmetric models of Grand Unification, however, are in favor [57, 58].

7.2 *Other renormalization schemes*

We briefly address the other renormalization schemes mentioned in the beginning of section 7. We restrict this discussion to parameter renormalization only.

- The G_μ-scheme:

The G_μ-scheme [30] with the parameters

$$\alpha,\ G_\mu,\ M_Z,\ m_f,\ M_H$$

treats G_μ as a basic parameter to be renormalized instead of the W mass. The counter terms, which appear in the bare quantities

$$e_0 = e + \delta e,\ \ M_Z^{0\,2} = M_Z^2 + \delta M_Z^2,\ \ G_\mu^0 = G_\mu + \delta G_\mu \tag{182}$$

after separating off the renormalized parameters, are determined by the on-shell conditions for e and M_Z as in the OS-scheme. The renormalization condition for G_μ, which replaces the on-shell condition for M_W, defines G_μ as the experimental Fermi constant, thus fixing the counter term δG_μ by the requirement of absorbing the 1-loop contribution to the μ-decay amplitude:

$$\frac{\delta G_\mu}{G_\mu} = -\frac{\Sigma^{WW}(0)}{M_W^2} - \frac{G_\mu M_W^2}{2\pi^2\sqrt{2}}\left[4\,(\Delta - \log\frac{M_W^2}{\mu^2}) + 6 + \frac{7-4s^2}{2s^2}\log c^2\right]. \tag{183}$$

The mixing angle is a derived quantity following from the exact relation between the bare quantities

$$s_0^2 c_0^2 = \frac{e_0^2}{4\sqrt{2}G_\mu^0 M_Z^{0\,2}} \tag{184}$$

by the one-loop expansion according to (182)

$$s^2 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}M_Z^2 G_\mu}}\right) \tag{185}$$

with the counter term in the decomposition $s_0^2 = s^2 + \delta s^2$

$$\delta s^2 = \frac{c^2 s^2}{c^2 - s^2}\left(2\frac{\delta e}{e} - \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta G_\mu}{G_\mu}\right). \tag{186}$$

The physical Wmass is obtained from the pole condition for the W-propagator as the solution of the equation

$$M_W^2 - m_W^2 - \delta M_W^2 + \mathrm{Re}\,\Sigma^{WW}(M_W^2) = 0 \tag{187}$$

with

$$\begin{aligned} m_W^2 &= \frac{M_Z^2}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}M_Z^2 G_\mu}}\right), \\ \frac{\delta M_W^2}{M_W^2} &= \frac{s^2}{c^2 - s^2}\left(\frac{\delta G_\mu}{G_\mu} - 2\frac{\delta e}{e} + \frac{c^2}{s^2}\frac{\delta M_Z^2}{M_Z^2}\right). \end{aligned} \tag{188}$$

Because of the large effects associated with the renormalization of e, s^2 is a bad approximation for the mixing angle in the perturbative expansion. The improved mixing angle

$$\tilde{s}^2 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\pi\alpha(M_Z^2)}{\sqrt{2}M_Z^2 G_\mu}}\right) \tag{189}$$

with $\alpha(M_Z^2)$ from Eq. (128) includes the resummed large contribution from the light fermions and is hence a better starting point for perturbative calculations. Making use of $\tilde{s}^2$, one has simultaneously to subtract $\Delta\alpha$ from the charge renormalization counter term and replace in Eq. (187)

$$2\frac{\delta e}{e} \to 2\frac{\delta e}{e} - \Delta\alpha\,. \tag{190}$$

- The low energy scheme:

The scheme with $e, G_\mu, \sin^2\theta_{\nu e}$ [31, 32] exclusively deals with parameters related to low energy experiments. The mixing angle $\sin^2\theta_{\nu e} = s_{\nu e}^2$ is treated as a fundamental parameter determined from ν-e scattering in terms of the ratio

$$R = \frac{\sigma(\nu_\mu e \to \nu_\mu e)}{\sigma(\bar{\nu_\mu} e \to \bar{\nu_\mu} e)} = \frac{(1-4s_{\nu e}^2)^2 + (1-4s_{\nu e}^2) + 1}{(1-4s_{\nu e}^2)^2 - (1-4s_{\nu e}^2) + 1}\,. \tag{191}$$

The renormalization of e and G_μ is the same as in the G_μ-scheme. For renormalizing $s_{\nu e}^2$, the counter term in $s_0^2 = s_{\nu e}^2 + \delta s_{\nu e}^2$ is fixed by the condition that $\delta s_{\nu e}^2$ absorbs the 1-loop contribution $\delta R^{(1)}$ to the ratio R:

$$R(s_{\nu e}^2 + \delta s_{\nu e}^2) + \delta R^{(1)} = R(s_{\nu e}^2)\,. \tag{192}$$

Taking the experimental result R_{exp} yields a numerical value for $s_{\nu e}^2$.

Both vector boson masses are derived quantities following from the pole conditions for W and Z:

$$\begin{aligned} M_W^2 - m_W^2 - \delta M_W^2 + \mathrm{Re}\,\Sigma^{WW}(M_W^2) &= 0, \\ M_Z^2 - m_Z^2 - \delta M_Z^2 + \mathrm{Re}\,\Sigma^{ZZ}(M_Z^2) &= 0 \end{aligned} \tag{193}$$

with

$$m_W^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu s_{\nu e}^2}, \quad m_Z^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu s_{\nu e}^2 c_{\nu e}^2} \tag{194}$$

and

$$\begin{aligned} \frac{\delta M_W^2}{M_W^2} &= 2\frac{\delta e}{e} - \frac{\delta G_\mu}{G_\mu} - \frac{\delta s_{\nu e}^2}{s_{\nu e}^2}, \\ \frac{\delta M_Z^2}{M_Z^2} &= 2\frac{\delta e}{e} - \frac{\delta G_\mu}{G_\mu} - \frac{c_{\nu e}^2 - s_{\nu e}^2}{c_{\nu e}^2}\frac{\delta s_{\nu e}^2}{s_{\nu e}^2}\,. \end{aligned} \tag{195}$$

A slightly modified version of this scheme was used in [32]. There the condition (192) was imposed in the $\overline{MS}$ renormalization prescription

$$R(\hat{s}^2_{\nu e}) + \delta\hat{R}^{(1)} = R_{exp}$$

yielding the $\overline{MS}$- version $\hat{s}^2_{\nu e}$ of the low energy mixing angle.

- The $*$ scheme:

The bare parameters of the Standard Model can be eliminated in a formally different way [36, 37] by introducing a set of 4 effective parameters

$$e_*(s),\ s^2_*(s),\ G_{\mu *}(s),\ \rho_*(s), \tag{196}$$

where in the minimal model only three are independent. These running parameters ($s = k^2$) contain the real parts of the self energies. They are arranged in such a way that the amplitude for a 4-fermion process with self energy corrections is obtained from the Born amplitude by the formal replacement

$$(e,\ s^2_W,\ G_\mu,\ \rho) \to (e_*,\ s^2_*,\ G_{\mu *},\ \rho_*), \tag{197}$$

supplemented by the corresponding imaginary parts. When the physical input is taken from the experimental data points α, G_μ, M_Z, the result for the 4-fermion scattering amplitudes is identical to that of the conventional on-shell scheme with self energy corrections after the 2-loop 1-particle irreducible leading contributions are built in.

In the following we give the relation between the conventional expressions of section 3.3 and the corresponding ones in terms of the $*$ -parameters. For a more detailed dicussion of the propagator corrections in the on-shell scheme we refer to the section on e^+e^- annihilation.

$$\begin{array}{rcl} \text{on-shell} & \leftrightarrow & * \\ \dfrac{e^2}{1+\mathrm{Re}\,\hat{\Pi}^\gamma(s)} & \leftrightarrow & e^2_*(s) \\ s^2_W - s_W c_W\,\mathrm{Re}\,\dfrac{\hat{\Pi}^{\gamma Z}(s)}{1+\hat{\Pi}^\gamma(s)} & \leftrightarrow & s^2_*(s) \\ \dfrac{e^2}{s^2_W}\cdot\dfrac{1}{s - M^2_W + \hat{\Sigma}^{WW}(s)} & \leftrightarrow & \dfrac{e^2_*}{s^2_*}\cdot\dfrac{1}{s - \frac{e^2_*}{s^2_*}\frac{1}{4\sqrt{2}G_{\mu *}} + i\sqrt{s}\,\Gamma_{*W}(s)} \\ \dfrac{e^2}{s^2_W c^2_W}\cdot\dfrac{1}{s - M^2_Z + \hat{\Sigma}^{Z}(s)} & \leftrightarrow & \dfrac{e^2_*}{s^2_* c^2_*}\cdot\dfrac{1}{s - \frac{e^2_*}{s^2_* c^2_*}\frac{1}{4\sqrt{2}G_{\mu *}\rho_*} + i\sqrt{s}\,\Gamma_{*Z}(s)} \end{array} \tag{198}$$

The quantities $\Gamma_{*Z}(s)$, $\Gamma_{*W}(s)$ correspond to the imaginary parts of the Z and W self energies. The relation to the physical Z width (and similar for W) is given by

$$\Gamma_Z = \frac{\Gamma_{*Z}(M_Z^2) + \Delta\Gamma_Z}{1+\kappa_*}$$

where $\Delta\Gamma_Z$ denotes the corrections to the Z width in $O(\alpha^2)$ not of the self energy type (vertex, QED and QCD corrections), discussed in the next chapter, and κ_* is determined by the residue of the Z propagator in (198):

$$s - \frac{e_*^2}{s_*^2 c_*^2}\frac{1}{4\sqrt{2}G_{\mu *}\rho_*} = (s - M_Z^2)\cdot(1+\kappa_*)\,.$$

The zero of the l.h.s. corresponds to the physical Z mass.

The * star arrangement as well as the on-shell one with resummation of the self energies contain higher order terms which are in general not gauge invariant. The leading terms, however, arise from light and heavy fermions which belong to the gauge invariant subclass of fermion loops, and the resummation yields the reducible higher order terms to all orders. The bosonic loop contributions on the other hand give gauge invariant results only when they are combined with vertex and box diagrams of the same order in a physical matrix element. They have always to be understood as expanded to one-loop order when appearing in formally higher order expressions. In the 't Hooft-Feynman gauge the numerical differences are irrelevant; in the unitary gauge, however, the individual contributions become divergent.

7.3 *Uncertainties of theoretical predictions*

In order to establish in a significant manner possibly small effects from unknown physics we have to know the uncertainties of our theoretical predictions which have to be confronted with the experiments.

The sources of uncertainties in theoretical predictions are the following:

- the experimental errors of the parameters used as an input. With the choice α, G_μ, and M_Z from LEP we can keep these errors as small as possible. The errors from this source are then determined by δM_Z since the errors of α and G_μ are negligibly small. For any of the mixing angles with $s_W^2, \hat{s}^2, s_f^2$

$$\frac{\delta s^2}{s^2} = \frac{2\,c^2}{c^2 - s^2}\frac{\delta M_Z}{M_Z} \tag{199}$$

  one finds

$$\delta s^2 \simeq 5\cdot 10^{-5}\,.$$

- the uncertainties from quark loop contributions to the radiative corrections . Here, we have to distinguish two cases: the uncertainties from the light quark contributions to $\Delta\alpha$ and the uncertainties from the heavy quark contributions to $\Delta\rho$. In both cases the uncertainties are due to strong interaction effects, which are not sufficiently under control theoretically. The problems are due to:
(i) the QCD parameters. The scale of α_s and the definition and scale of quark masses to be used in the calculation of a particular quantity are quite ambiguous in many cases.
(ii) the bad convergence and/or breakdown of perturbative QCD. In particular at low q^2 and in the resonance regions theoretically poorly known nonperturbative effects are non-negligible.

The theoretical problems with the hadronic contributions of the 5 known light quarks to $\Delta\alpha$ can be circumvented by using the experimental e^+e^--annihilation cross-section $\sigma_{tot}(e^+e^- \to \gamma^* \to hadrons)$. The error [44]

$$\delta(\Delta\alpha) = \pm 0.0009$$

is dominated by the large experimental errors in the continuum contributions to $\sigma_{tot}(e^+e^- \to \gamma^* \to hadrons)$ below the Υ threshold, and can be improved only by more precise measurements of hadron production in e^+e^--annihilation in the corresponding low energy region. This uncertainty leads to an error in the W-mass prediction

$$\frac{\delta M_W}{M_W} = \frac{s_W^2}{c_W^2 - s_W^2} \frac{\delta(\Delta r)}{2(1-\Delta r)}$$

of $\delta M_W = 17$ MeV and $\delta \sin^2\theta = 0.0003$ in the prediction of the various weak mixing parameters $s_W^2, \hat{s}^2, s_f^2$.

The contribution to $\Delta\rho$ from quark doublets with large mass splitting exhibits large QCD corrections of the weak current quark loops. For a heavy top one finds

$$\Delta\rho = \frac{\sqrt{2}G_\mu}{16\pi^2} 3m_t^2 K_{QCD} + \cdots$$

with

$$K_{QCD} = 1 - \frac{2\pi^2+6}{9}\frac{\alpha_s}{\pi}$$

for asymptotically large m_t [52]. The corrections obtained are not well determined numerically because it remains unclear which scale should be chosen for α_s. Also the ambiguity in the definition of m_t has not been taken into account.

Again, the problem can be controlled better by using dispersion relations. In this approach, the remaining uncertainties in Δr

$$\delta(\Delta r)_{QCD} \simeq \begin{cases} 0.0005 & m_t < 150GeV \\ 0.0015 \cdot (m_t/250GeV)^2 & m_t > 150GeV \end{cases} \tag{200}$$

have been estimated in [59]. In the heavy top region, where the errors of Δr and $\Delta\rho$ are correlated by $\delta(\Delta r) \simeq c_W^2/s_W^2 \delta(\Delta\rho)$, the uncertainties in the NC couplings in Eq. (134) can be estimated in terms of $\delta\Delta\rho$. The error of the normalization turns out to be smaller than $5 \cdot 10^{-4}$, and for the mixing angle one finds

$$\delta s^2 < 0.00015\,.$$

- the uncertainties from omission of higher order effects. After resummation of the leading terms, how large are the omitted higher order effects? Since a complete two-loop calculation has not been done, we only can guess how large such effects could be. In the calculation of Δr the difference is given, in the approximation we consider, by using different parameters in the evaluation of $\Delta r_{remainder}$ defined in Eq. (140). A supposedly conservative estimate of the error made by omitting the higher order effects has been given in [60].

$$\delta(\Delta r)_{higher-order} = \pm 0.001 \tag{201}$$

which can be added quadratically to the hadronic errors. Explicit comparisons between OS and $\overline{MS}$ calculations [35] as well as between different versions of the OS scheme [61] for the Z resonance observables have shown to be well below the experimental uncertainties. The typical size of the theoretical uncertainty of improved one-loop calculations is thus around 0.001.

8 Extension to larger theories

We want to conclude this chapter with an outlook on renormalizable generalizations of the minimal model and their effect on electroweak observables. Extended models can be classified in terms of the following categories:

(i) extensions within the minimal gauge group SU(2)×U(1) with $\rho_{tree} = 1$

(ii) extensions within SU(2)×U(1) with $\rho_{tree} \neq 1$

(iii) extensions with larger gauge groups SU(2)×U(1)×G and respective extra gauge bosons.

Extensions of the class (i) are, for example, models with additional (sequential) fermion doublets, more Higgs doublets, and the minimal supersymmetric version of the Standard Model.

8.1 *Paramatrization of self energy corrections*

If "new physics" would be present in form of new particles which couple to the gauge bosons but not directly to the external fermions in a 4-fermion process, the formulae for the self energies in section 3.3 are general enough that those effects can be built in by calculating the additonal loop diagrams.

In order to have a description which is as far as possible independent of the special type of extra heavy particles, it is convenient to introduce a parametrization of the radiative corrections from the vector boson self-energies in terms of the static ρ-parameter

$$\Delta\rho(0) = \frac{\Sigma^{ZZ}(0)}{M_Z^2} - \frac{\Sigma^{WW}(0)}{M_W^2} - 2\frac{s_W}{c_W}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} \tag{202}$$

and the combinations

$$\begin{aligned} \Delta_1 &= \frac{1}{s_W}\Pi^{3\gamma}(M_Z^2) - \Pi^{33}(M_Z^2) \\ \Delta_2 &= \Pi^{33}(M_Z^2) - \Pi^{WW}(M_W^2) \\ \Delta\alpha &= \Pi^{\gamma\gamma}(0) - \Pi^{\gamma\gamma}(M_Z^2)\,. \end{aligned} \tag{203}$$

The quantities in Eq. (203) are the isospin components of the self-energies

$$\begin{aligned} \Sigma^{\gamma Z} &= -\frac{1}{c_W}\left(\Sigma^{3\gamma} - s_W^2\Sigma^{\gamma\gamma}\right) \\ \Sigma^{ZZ} &= \frac{1}{c_W^2}\left(\Sigma^{33} - 2s_W\Sigma^{3\gamma} + s_W^2\Sigma^{\gamma\gamma}\right) \end{aligned} \tag{204}$$

in the expansions

$$\mathrm{Re}\,\Sigma^{ij}(k^2) = \Sigma^{ij}(0) + k^2\Pi^{ij}(k^2)\,. \tag{205}$$

The Δ-notation above has been introduced in [62]. Several other conventions are used in the literature:

- The S, T, U parameters of [63] are related to (203) by

$$S = \frac{4s_W^2}{\alpha}\Delta_1, \;\; T = \frac{1}{\alpha}\Delta\rho(0), \;\; U = \frac{4s_W^2}{\alpha}\Delta_2, \tag{206}$$

- the ϵ-parameters of [64] by

$$\epsilon_1 = \Delta\rho, \;\; \epsilon_2 = -\Delta_2, \;\; \epsilon_3 = \Delta_1, \tag{207}$$

- the h-parameters of [65] by

$$h_V = \frac{1}{\alpha}\Delta\rho(0), \quad h_{AZ} = \frac{4\pi}{\sqrt{2}G_\mu M_W^2}\Delta_1, \quad h_{AW} = h_{AZ} + \frac{4\pi}{\sqrt{2}G_\mu M_W^2}\Delta_2, \tag{208}$$

- and the parameters of [67] by

$$\Delta_\rho(0) = \frac{1}{4\sqrt{2}G_\mu}\Delta\rho, \quad \Delta_3 = -\frac{1}{4\sqrt{2}G_\mu c_W^2}\Delta_1, \quad \Delta_\pm = c_W^2\Delta_3 - \frac{1}{4\sqrt{2}G_\mu}\Delta_2. \tag{209}$$

The combinations (203) of self energies contribute in a universal way to the electroweak parameters (the residual corrections not from self-energies are dropped since they are identical to the Standard Model ones):

1. the $M_W - M_Z$ correlation in terms of Δr:

$$\Delta r = \Delta\alpha - \frac{c_W^2}{s_W^2}\Delta\rho(0) - \frac{c_W^2 - s_W^2}{s_W^2}\Delta_2 + 2\Delta_1 \tag{210}$$

2. the normalization of the NC couplings at M_Z^2

$$\Delta\rho_f = \Delta\rho(0) + \Delta_Z \tag{211}$$

   where the extra quantity

$$\Delta_Z = M_Z^2\,\frac{d\,\Pi^{ZZ}}{dk^2}(M_Z^2)$$

   in (211) is from the residue of the Z propagator at the peak. Heavy particles decouple from Δ_Z.

3. the effective mixing angles

$$s_f^2 = (1 + \Delta\kappa')\,\tilde{s}^2, \quad \tilde{s}^2 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\pi\alpha(M_Z^2)}{\sqrt{2}G_\mu M_Z^2}}\right), \tag{212}$$

   with

$$\Delta\kappa' = -\frac{c_W^2}{c_W^2 - s_W^2}\Delta\rho(0) + \frac{\Delta_1}{c_W^2 - s_W^2}. \tag{213}$$

The finite combinations of self energies (202) and (203) are of practical interest since they can be extracted from precision data in a fairly model independent way. An experimental observable particular sensitive to Δ_1 is the weak charge Q_W which determines the atomic parity violation in Cesium [66]

$$Q_W = -73.20 \pm 0.13 - 0.82\Delta\rho(0) - 102\Delta_1 \tag{214}$$

being almost independent of $\Delta\rho(0)$.

The theoretical interest in the Δ's is based on their selective sensitivity to different kinds of new physics.

- $\Delta\alpha$ gets contributions only from light charged particles whereas heavy objects decouple.

- $\Delta\rho(0)$ is a measure of the violation of the custodial $SU(2)$ symmetry. It is sensitive to particles with large mass splittings in multiplets. As an example, we have already encountered fermion doublets with different masses, see Eq. (129). Another example are the Higgs bosons of a 2-Higgs doublet model [69, 70, 71, 72] with masses M_{H^+}, M_h, M_H, M_A and mixing angles β, α for the charged $H^\pm$ and the neutral h^0, H^0, A^0 Higgs bosons, yielding

$$\begin{aligned}\Delta\rho(0) &= \frac{G_\mu}{8\pi^2\sqrt{2}}\left[\sin^2(\alpha-\beta)F(M_{H^+}^2, M_A^2, M_H^2)\right.\\ &\quad \left.+\cos^2(\alpha-\beta)F(M_{H^+}^2, M_A^2, M_h^2)\right] \qquad (215)\end{aligned}$$

with

$$F(x,y,z) = x + \frac{yz}{y-z}\log\frac{y}{z} - \frac{xy}{x-y}\log\frac{x}{y} - \frac{xz}{x-z}\log\frac{x}{z}\,.$$

For either $M_{H^+} \gg M_{neutral}$ or vice versa one finds a positive contribution

$$\Delta\rho(0) \simeq \frac{G_\mu M_{H^+}^2}{8\pi^2\sqrt{2}} \quad \text{or} \quad \frac{G_\mu M_{neutral}^2}{8\pi^2\sqrt{2}} > 0\,. \qquad (216)$$

Also a negative contribution

$$\Delta\rho(0) < 0 \quad \text{for} \quad M_{h,H} < M_{H^+} < M_A \quad \text{and} \quad M_A < M_{H^+} < M_{h,H}$$

is possible in the unconstrained 2-doublet model.

- Δ_1 is sensitive to chiral symmetry breaking by masses. In particular, a doublet of mass degenerate heavy fermions yields a contribution

$$\Delta_1 = N_C^f\,\frac{G_\mu M_W^2}{12\pi^2\sqrt{2}}, \qquad (217)$$

whereas the contribution of degenerate heavy fermions to $\Delta\rho(0)$ is zero. Hence, Δ_1 can directly count the number N_{deg} of mass degenerate fermion doublets:

$$\Delta_1^f = 4.5\cdot 10^{-4}\cdot N_{deg}.$$

Δ_1 also gets sizeable contributions from models with a large number of additional fermions like in technicolor models. For example, $\Delta_1 \simeq 0.017$ for $N_{TC} = 4$ and one family of technifermions [63, 68].

8.2 *Models with* $\rho_{tree} \neq 1$

One of the basic relations of the minimal Standard Model is the tree level correlation between the vector boson masses and the electroweak mixing angle

$$\rho_{tree} = \frac{M_W^2}{M_Z^2 \sin^2\theta_W} = 1\,.$$

Many extensions of the minimal model, like those discussed in the previous section, preserve this feature.

The formulation of the electroweak theory in terms of a local gauge theory requires at least a single scalar doublet for breaking the electroweak symmetry SU(2)×U(1) → U(1)$_{em}$. In contrast to the fermion and vector boson part, very little is known empirically about the scalar sector. Without the assumption of minimality, quite a lot of options are at our disposal, including more complicated multiplets of Higgs fields. In general models the tree level ρ-parameter $\rho_{tree} = \rho_0$ is determined by

$$\rho_0 = \frac{\sum_i v_i^2[I_i(I_i+1) - I_{3i}^2]}{2\sum_i v_i^2 I_{3i}^2}$$

where v_i, I_{3i} are the vacuum expectation values and third isospin component of the neutral component of the i-th Higgs multiplet in the representation with isospin I_i. The presence of at least a triplet of Higgs fields gives rise to $\rho_0 \neq 1$. As a consequence, the tree level relations between the electroweak parameters have to be generalized according to

$$\sin^2\theta_W \rightarrow s_\theta^2 = 1 - \frac{M_W^2}{\rho_0 M_Z^2} \tag{218}$$

and

$$\frac{G_\mu}{\sqrt{2}} = \frac{e^2}{8 s_\theta^2 M_W^2} = \frac{e^2}{8 s_\theta^2 c_\theta^2 \rho_0 M_Z^2} \tag{219}$$

Writing $\rho_0 = (1 - \Delta\rho_0)^{-1}$, we obtain for the mixing angle:

$$s_\theta^2 = 1 - \frac{M_W^2}{M_Z^2} + \frac{M_W^2}{M_Z^2}\Delta\rho \equiv s_W^2 + c_W^2 \Delta\rho_0\,, \tag{220}$$

for the overall normalization factor in the NC vertex:

$$\frac{e}{2 s_\theta c_\theta} = \left(\sqrt{2} G_\mu M_Z^2 \rho_0\right)^{1/2}\,, \tag{221}$$

and for the $M_W - M_Z$ interdependence:

$$M_W^2\left(1 - \frac{M_W^2}{\rho_0 M_Z^2}\right) = \frac{e^2}{4\sqrt{2} G_\mu}\,, \tag{222}$$

in complete analogy to what we have found from the top quark loops.

At the level of radiative corrections, a small $\Delta\rho_0$ may be included by

$$\Delta r \rightarrow \Delta r - \frac{c_W^2}{s_W^2}\Delta\rho_0 \tag{223}$$

for the M_W-M_Z correlation, and

$$\rho_f \rightarrow \rho_f + \Delta\rho_0, \quad s_f^2 \rightarrow s_f^2 + c_W^2\Delta\rho_0 \tag{224}$$

for the normalization and the effective mixing angles of the Zff couplings.

A complete discussion of radiative corrections requires not only the calculation of the extra loop diagrams from the non-standard Higgs sector but also an extension of the renormalization procedure [73, 74]. Since M_W, M_Z and $\sin^2\theta_W$ (or ρ_0, eqivalently) are now independent parameters, one extra renormalization condition is required. A natural condition would be to define the mixing angle for electrons s_e^2 in terms of the ratio of the dressed coupling constants at the Z peak

$$\frac{g_V^e}{g_A^e} =: 1 - 4s_e^2$$

which is measureable in terms of the left-right or the forward-backward asymmetries. This fixes the counter term for s_e^2 by

$$\frac{\delta s_e^2}{s_e^2} = \frac{c_e}{s_e}\frac{\mathrm{Re}\,\Sigma^{\gamma Z}(M_Z^2)}{M_Z^2} + 2\frac{c_e}{s_e}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} + \Delta\kappa_e \tag{225}$$

with the finite part $\Delta\kappa_e$ of the electron-Z vertex correction. The counter terms for the other parameters α, M_Z are treated as usual. With this input, we obtain a renormalized ρ-parameter and the corresponding counter term for the bare ρ-parameter $\rho_0^b = \rho + \delta\rho$ as follows:

$$\begin{aligned} \rho &= \frac{M_W^2}{M_Z^2 c_e^2}, \\ \frac{\delta\rho}{\rho} &= \frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2} + \frac{\delta s_e^2}{c_e^2}. \end{aligned} \tag{226}$$

Other derived quantities are:

- The relation between M_W and G_μ:

$$M_W^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu s_e^2} \cdot \frac{1}{1-\Delta r} \tag{227}$$

with

$$\Delta r = \frac{\Sigma^{WW}(0) - \delta M_W^2}{M_W^2} + \Pi^\gamma(0) - \frac{\delta s_e^2}{s_e^2} + 2\frac{c_e}{s_e}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2} + \delta_{VB}\,. \tag{228}$$

- The normalization of the Zff couplings at 1-loop:

$$\frac{e^2}{4s_e^2c_e^2}\left[1+\Pi^{\gamma}(0)-\frac{c_e^2-s_e^2}{c_e^2}\frac{\delta s_e^2}{s_e^2}+2\frac{c_e^2-s_e^2}{c_es_e}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2}+\Delta\rho_f\right] \tag{229}$$

$$= \sqrt{2}G_\mu M_Z^2\rho\left[1-\frac{\Sigma^{WW}(0)-\delta M_W^2}{M_W^2}+\frac{\delta s_e^2}{c_e^2}-2\frac{s_e}{c_e}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2}-\delta_{VB}+\Delta\rho_f\right]$$

where $\Delta\rho_f$ denotes the finite part of the Zff vertex correction.

- The effective mixing angles of the Zff couplings:

$$s_f^2 = s_e^2(1-\Delta\kappa_e+\Delta\kappa_f).$$

These relations predict the Z boson couplings, M_W and ρ in terms of the data points $\alpha, G_\mu, M_Z, s_e^2$. By this procedure, the leading m_t^2-dependence of the self energy corrections to theoretical predictions is absorbed into the renormalized ρ-parameter, leaving a $\sim \log m_t/M_Z$ term as an observable effect. For the Zbb-vertex, an additional m_t^2 dependence is found in the non-universal vertex corrections $\Delta\rho_b$ and $\Delta\kappa_b$. This makes observables containing this vertex the most sensitive top indicators in the class of models with $\rho_{tree}\neq 1$.

In the minimal Standard Model, the quantity equivalent to (226) can be calculated in terms of the data points α, G_μ, M_Z and the parameters m_t, M_H. With the experimental constraints from M_W in section 5.3 and and $s_e^2 = 0.2328 \pm 0.0007$ from LEP data [1, 4] we obtain

$$\rho_{SM} = 1.0069 \pm 0.0040\,. \tag{230}$$

In the extended models we can calculate ρ from

$$\rho = \frac{\pi\alpha}{\sqrt{2}G_\mu M_Z^2 s_e^2 c_e^2}\cdot\frac{1}{1-\Delta r} \tag{231}$$

in terms of the input data $\alpha, G_\mu, M_Z, s_e^2$ together with m_t and the parameters of the Higgs sector. Such a complete calculation, however, does not exist as yet. Instead, we can get a value for ρ from directly using the data on M_W^2/M_Z^2 and $s_e^2 = 0.2324 \pm 0.0011$ from forward-backward asymmetries at LEP [1, 4] yielding

$$\rho = 1.0064 \pm 0.0069\,. \tag{232}$$

The difference $\rho - \rho_{SM}$ can be interpreted as a measure for a deviating tree level structure. The data imply that it is compatible with zero.

8.3 *Extra Z bosons*

The existence of additional vector bosons is predicted by GUT models based on groups bigger than SU(5), like E_6 and $SO(10)$, by models with symmetry breaking in terms of a strongly interacting sector, and composite scenarios. Typical examples of extended gauge symmetries are the $SU(2) \times U(1) \times U(1)_{\chi,\psi,\eta}$ models following from E_6 unification, or LR-symmetric models. In the following we consider only models with an extra $U(1)$.

The mixing between the mathematical states Z_0 of the minimal gauge group and Z_0' of an extra hypercharge form the physical mass eigenstates Z, Z', where the lighter Z is identified with the resonance at LEP. The mass eigenstates are obtained by a rotation

$$\begin{aligned} Z &= \cos\theta_M \, Z_0 + \sin\theta_M \, Z_0' \\ Z' &= -\sin\theta_M \, Z_0 + \cos\theta_M \, Z_0' \end{aligned} \qquad (233)$$

with a mixing angle θ_M related to the mass eigenvalues by

$$\tan^2\theta_M = \frac{M_{Z_0}^2 - M_Z^2}{M_{Z'}^2 - M_{Z_0}^2}, \quad M_{Z_0}^2 = \cos^2\theta_M \, M_Z^2 + \sin^2\theta_M \, M_{Z'}^2 \,. \qquad (234)$$

$M_{Z_0}^2$ denotes the nominal mass of Z_0. In constrained models with the Higgs fields in doublets and singlets only, the usual Standard Model relation holds

$$\sin^2\theta_W = 1 - \frac{M_W^2}{M_{Z_0}^2}$$

between the masses and the mixing angle in the Lagrangian

$$\mathcal{L}_{NC} = \frac{g_2}{\cos\theta_W} J_{Z_0}^\mu \, Z_0^\mu \; + \; g' \, J_{Z_0'}^\mu \, Z_0'^{\mu} \qquad (235)$$

with

$$J_{Z_0}^\mu = J_L^\mu - \sin^2\theta_W \, J_{em}^\mu \,.$$

It is convenient to introduce the quantity

$$s_W^2 = 1 - \frac{M_W^2}{M_Z^2}, \quad c_W^2 = 1 - s_W^2 \qquad (236)$$

with the physical mass of the lower eigenstate. For small mixing angles θ_M we have the following relation:

$$\sin^2\theta_W = s_W^2 + c_W^2 \Delta\rho_{Z'} \qquad (237)$$

with

$$\Delta\rho_{Z'} = \sin^2\theta_M \left(\frac{M_{Z'}^2}{M_Z^2} - 1 \right) \,. \qquad (238)$$

The W mass is obtained from

$$M_W^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu \sin^2\theta_W(1-\Delta r)}$$

after the substitution (237):

$$M_W^2 = \frac{M_Z^2}{2}\left(1+\sqrt{1-\frac{\pi\alpha}{\sqrt{2}G_\mu M_Z^2\rho_{Z'}(1-\Delta r)}}\right) \tag{239}$$

with $\rho_{Z'} = (1-\Delta\rho_{Z'})^{-1}$. Formally, $\rho_{Z'}$ appears as a non-standard tree level ρ-parameter. In all present practical applications the radiative correction Δr was approximated by the standard model correction.

The mass mixing has two implications for the NC couplings of the Z boson:

- $\Delta\rho_{Z'}$ contributes to the overall normalization by a factor

  $$\rho_{Z'}^{1/2} \simeq 1 + \frac{1}{2}\Delta\rho_{Z'}$$

  and to the mixing angle by a shift

  $$s_W^2 \to s_W^2 + c_W^2\Delta\rho_{Z'}\,.$$

  Both effects are universal, parametrized by $M_{Z'}$ and the mixing angle θ_M in a model independent way,

- A non-universal contribution is present as the second term in the vertex

  $$\begin{aligned}(Zff) &= \cos\theta_M(Z_0ff) + \sin\theta_M(Z_0'ff)\\ &\simeq (Z_0ff) + \theta_M(Z_0'ff)\,.\end{aligned}$$

  It depends on the classification of the fermions under the extra hypercharge and is strongly model dependent.

Complete 1-loop calculations are not available as yet. The present standard approach consists in the implementation of the standard model corrections to the Z_0 parts of the coupling constants in terms of the form factors ρ_f for the normalization and κ_f for the effective mixing angles

$$s_W^2 \to s_f^2 = \kappa_f s_W^2.$$

In this approach the effective Zff vector and axial vector couplings read:

$$\begin{aligned}v_Z^f &= \left[\sqrt{2}G_\mu M_Z^2\rho_f(1+\Delta\rho_{Z'})\right]^{1/2}\left[I_3^f - 2Q_f(\kappa_f s_W^2 + c_W^2\Delta\rho_{Z'})\right]\\ &\quad + \sin\theta_M\, v_{Z_0'}^f\,,\\ a_Z^f &= \left[\sqrt{2}G_\mu M_Z^2\rho_f\right]^{1/2} I_3^f + \sin\theta_M\, a_{Z_0'}^f\,.\end{aligned} \tag{240}$$

The quantities $a^f_{Z'_0}$ $v^f_{Z'_0}$ denote the extra $U(1)$ couplings between the fermion f and the Z'_0.

From an analysis of the electroweak precision data the mixing angle is constrained typically to $|\theta_M| < 0.01$, not very much dependent on the specification of the model [75, 76]. An example is shown in Figure 9.

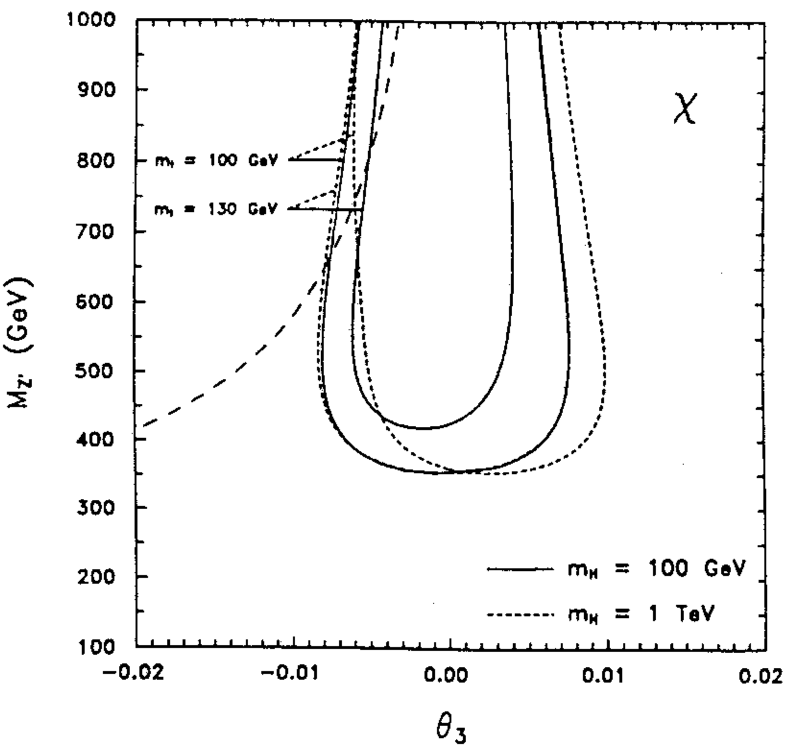


Fig. 9: *90% C.L. contours for mass and mixing angle ($\theta_3 = \theta_M$) of the extra Z' in the $SU(2) \times U(1) \times U(1)_\chi$ model, from [76]*

References:

1. R. Tanaka, talk presented at the *XXVI International Conference on High Energy Physics*, Dallas 1992. Saclay preprint X-LPNHE/92-3, to appear in the proceedings
2. H. Plothow-Besch, in: *Proceeedings of the LP-HEP 91 Conference*, Geneva 1991, eds. S. Hegarty, K. Potter, E. Quercigh, World Scientific, Singapore 1992
3. G.L. Fogli and D. Haidt, *Z. Phys.* **C40** (1988) 379;
   CDHS Collaboration, H. Abramowicz et al., *Phys. Rev. Lett.* **B57** (1986) 298;
   A. Blondel et al, *Z. Phys.* **C45** (1990) 361;
   CHARM Collaboration, J.V. Allaby et al., *Phys. Lett.* **B177** (1987) 446;
   Z. Phys. 36 (1987) 611;
   CHARM-II Collaboration, D. Geiregat et al., *Phys. Lett.* **B 247** (1990) 131;
   Phys. Lett. **B259** (1991) 499;
   CCFR Collaboration, talk presented by T. Bolton at the *XXVI International Conference on High Energy Physics*, Dallas 1992, to appear in the proceedings
4. G. Rolandi, plenary talk presented at the *XXVI International Conference on High Energy Physics*, Dallas 1992. CERN-PPE/92-175 (1992), to appear in the proceedings
5. S.L. Glashow, *Nucl. Phys.* **B22** (1961) 579;
   S. Weinberg, *Phys. Rev. Lett.* **19** (1967) 1264;
   A. Salam, in: *Proceedings of the 8th Nobel Symposium*, p. 367, ed. N. Svartholm, Almqvist and Wiksell, Stockholm 1968
6. S.L. Glashow, I. Iliopoulos, L. Maiani, *Phys. Rev.* **D2** (1970) 1285;
7. N. Cabibbo, *Phys. Rev. Lett.* **10** (1963) 531;
   M. Kobayashi, K. Maskawa, *Prog. Theor. Phys.* **49** (1973) 652
8. H.Y. Han, Y. Nambu, *Phys. Rev.* **139** (1965) 1006;
   C. Bouchiat, I. Iliopoulos, Ph. Meyer, *Phys. Lett.* **B138** (1972) 652
9. G. 't Hooft, *Nucl. Phys.* **B33** (1971) 173; *Nucl. Phys.* **B35** (1971) 167
10. CDF Collaboration, M. Contreras, *Proceeedings of the LP-HEP 91 Conference*, Geneva 1991, eds. S. Hegarty, K. Potter, E. Quercigh, World Scientific, Singapore 1992
11. D0 Collaboration, talk presented by S. Protopopescu at the *XXVIIIth Rencontres de Moriond: Electroweak Interactions and Unified Theories*, Les Arcs 1993;
    CDF Collaboration, talk presented by B. Harral at the *XXVIIIth Rencontres de Moriond: Electroweak Interactions and Unified Theories*, Les Arcs 1993
12. T. Appelquist, J. Carazzone, *Phys. Rev.* **D11** (1975) 2856
13. Particle data group, *Phys. Rev.* **D45** (1992) Number 11, Part II
14. L.D. Faddeev, V.N. Popov, *Phys. Lett.* **B25** (1967) 29

15. M. Böhm, W. Hollik, H. Spiesberger, *Fortschr. Phys.* **34** (1986) 687
16. A. Denner, T. Sack, *Nucl. Phys.* **B347** (1990) 203
17. G. Passarino, in: *Proceeedings of the LP-HEP 91 Conference*, Geneva 1991, eds. S. Hegarty, K. Potter, E. Quercigh, World Scientific, Singapore 1992
18. G. 't Hooft, *Nucl. Phys.* **B61** (1973) 455; *Nucl. Phys.* **B62** (1973) 444
19. D.A. Ross, J.C. Taylor, *Nucl. Phys.* **B51** (1973) 25
20. G. Passarino, M. Veltman, *Nucl. Phys.* **B160** (1979) 151
21. M. Consoli, *Nucl. Phys.* **B160** (1979) 208
22. A. Sirlin, *Phys. Rev.* **D22** (1980) 971;
W. J. Marciano, A. Sirlin, *Phys. Rev.* **22** (1980) 2695;
A. Sirlin, W. J. Marciano, *Nucl. Phys.* **189** (1981) 442
23. D.Yu. Bardin, P.Ch. Christova, O.M. Fedorenko, *Nucl. Phys.* **B175** (1980) 435; *Nucl. Phys.* **B197** (1982)1
24. J. Fleischer, F. Jegerlehner, *Phys. Rev.* **D23** (1981) 2001
25. K.I. Aoki, Z. Hioki, R. Kawabe, M. Konuma, T. Muta, *Suppl. Prog. Theor. Phys.* **73** (1982) 1;
Z. Hioki, *Phys. Rev. Lett.* **65** (1990) 683, E:1692; *Z. Phys.* **C49** (1991) 287
26. M. Consoli, S. LoPresti, L. Maiani, *Nucl. Phys.* **B223** (1983) 474
27. D.Yu. Bardin, M.S. Bilenky, G.V. Mithselmakher, T. Riemann, M. Sachwitz, *Z. Phys.* **C44** (1989) 493
28. W. Hollik, *Fortschr. Phys.* **38** (1990) 165
29. M. Consoli, W. Hollik, F. Jegerlehner, in: *Z Physics at LEP 1*, eds. G. Altarelli, R. Kleiss and C. Verzegnassi, CERN 89-08 (1989)
30. G. Passarino, R. Pittau, *Phys. Lett.* **B228** (1989) 89;
V.A. Novikov, L.B. Okun, M.I. Vysotsky, CERN-TH.6538/92 (1992)
31. M. Veltman, *Phys. Lett.* **B91** (1980) 95;
M. Green, M. Veltman, *Nucl. Phys.* **B169** (1980) 137, E: *Nucl. Phys.* **B175** (1980) 547;
F. Antonelli, M. Consoli, G. Corbo, *Phys. Lett.* **B91** (1980) 90;
F. Antonelli, M. Consoli, G. Corbo, O. Pellegrino, *Nucl. Phys.* **B183** (1981) 195
32. G. Passarino, M. Veltman, *Phys. Lett.* **B237** (1990) 537
33. W.J. Marciano, A. Sirlin, *Phys. Rev. Lett.* **46** (1981) 163;
A. Sirlin, *Phys. Lett.* **B232** (1989) 123
34. G. Degrassi, S. Fanchiotti, A. Sirlin, *Nucl. Phys.* **B351** (1991) 49
35. G. Degrassi, A. Sirlin, *Nucl. Phys.* **B352** (1991) 342
36. D.C. Kennedy, B.W. Lynn, *Nucl. Phys.* **B322** (1989) 1
37. M. Kuroda, G. Moultaka, D. Schildknecht, *Nucl. Phys.* **B350** (1991) 25
38. D.Yu. Bardin, B.M. Vilensky, P.Ch. Christova, *Sov. J. Nucl. Phys.* **53** (1991)

152;
A. Dabelstein, W. Hollik, *Z. Phys.* **C53** (1992) 25;
B.A. Kniehl, *Nucl. Phys.* **B376** (1992) 3

39. J.C. Ward, *Phys. Rev.* **78** (1950) 1824
40. G. 't Hooft, M. Veltman, *Nucl. Phys.* **B135** (1979) 365
41. C. Bollini, J. Giambiagi *Nuovo Cim.* **12B** (1972) 20;
J. Ashmore, *Nuovo Cim. Lett.* **4** (1972) 289;
G. 't Hooft, M. Veltman, *Nucl. Phys.* **B44** (1972) 189
42. P. Breitenlohner, D. Maison, *Comm. Math. Phys.* **52** (1977) 11, 39, 55
43. A. Denner, U. Nierste, R. Scharf, *Nucl. Phys.* **B367** (1991) 637
44. H. Burkhardt, F. Jegerlehner, G. Penso, C. Verzegnassi, *Z. Phys.* **C43** (1989) 497;
F. Jegerlehner, preprint Paul-Scherrer-Institut, PSI-PR-91-16 (1991), in: *Progress in Particle and Nuclear Physics*, ed. A. Fässler, Pergamon Press, Oxford, U.K.
45. M. Veltman, *Nucl. Phys.* **B123** (1977) 89;
M.S. Chanowitz, M.A. Furman, I. Hinchliffe, *Phys. Lett.* **B78** (1978) 285
46. M. Veltman, *Acta Phys. Polon.* **B8** (1977) 475.
47. R.E Behrends, R.J. Finkelnstein, A. Sirlin, *Phys. Rev.* **101** (1956) 866;
T. Kinoshita, A. Sirlin, *Phys. Rev.* **113** (1959) 1652
48. W.J. Marciano, *Phys. Rev.* **D20** (1979) 274
49. M. Consoli, W. Hollik, F. Jegerlehner, *Phys. Lett.* **B227** (1989) 167.
50. J.J. van der Bij, F. Hoogeveen, *Nucl. Phys.* **B283** (1987) 477
51. R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci, A. Vicere, *Phys. Lett.* **B288** (1992) 95; CERN-TH.6713/92 (1992)
52. A. Djouadi, C. Verzegnassi, *Phys. Lett.* **B195** (1987) 265;
A. Djouadi, *Nuovo Cim.* **100A** (1988) 357;
D. Yu. Bardin, A.V. Chizhov, Dubna preprint E2-89-525 (1989);
B.A. Kniehl, J.H. Kühn, R.G. Stuart, *Phys. Lett.* **B214** (1988) 621;
B.A. Kniehl, *Nucl. Phys.* **B347** (1990) 86;
F. Halzen, B.A. Kniehl, *Nucl. Phys.* **B353** (1991) 567
53. B.A. Kniehl, A. Sirlin, DESY 92-102 (1992);
S. Fanchiotti, B.A. Kniehl, A. Sirlin, CERN-TH.6749/92 (1992)
54. A. Sirlin, *Phys. Rev.* **D29** (1984) 89
55. S. Fanchiotti, A. Sirlin, New York University preprint NYU-Th-91/02/04 (1991), in: *Beg Memorial Volume*, eds. A. Ali and P. Hoodbhoy, World Scientific, Singapore 1991
56. A.A. Akhundov, D.Yu. Bardin, T. Riemann, *Nucl. Phys.* **B276** (1986) 1;
W. Beenakker, W. Hollik, *Z. Phys.* **C40** (1988) 141;
J. Bernabeu, A. Pich, A. Santamaria, *Phys. Lett.* **B200** (1988) 569; CERN-

TH.5931/90 (1990)

57. P. Langacker, M. Luo, *Phys. Rev.* **D44** (1991) 817
58. U. Amaldi, W. de Boer, H. Fürstenau, *Phys. Lett.* **B260** (1991) 447;
J. Ellis, S. Kelley, D.V. Nanopoulos, *Phys. Lett.* **B260** (1991) 131; *Phys. Lett.* **B287** (1992) 725;
G.G. Ross, R.G. Roberts, *Nucl. Phys.* **B377** (1992) 571
59. B.A. Kniehl, J.H. Kühn, R.G. Stuart, in: *Polarization at LEP*, eds. G. Alexander et al., CERN 88-06 (1988)
60. W. Hollik, H.J. Timme, *Z. Phys.* **C32** (1986) 125;
F. Jegerlehner, *Z. Phys.* **C32** (1986) 425
61. D.Yu. Bardin, W. Hollik, T. Riemann, *Z. Phys.* **C49** (1991) 485
62. G. Burgers, F. Jegerlehner, in: *Z Physics at LEP 1*, eds. G. Altarelli, R. Kleiss and C. Verzegnassi, CERN 89-08 (1989)
63. M.E. Peskin, T. Takeuchi, *Phys. Rev. Lett.* **65** (1990) 964
64. G. Altarelli, R. Barbieri, *Phys. Lett.* **B253** (1991) 161;
G. Altarelli, R. Barbieri, S. Jadach, *Nucl. Phys.* **B269** (1992) 3; E: *Nucl. Phys.* **B276** (1992) 444
65. D.C. Kennedy, P. Langacker, *Phys. Rev. Lett.* **65** (1990) 2967
66. W.J. Marciano, J.L. Rosner, *Phys. Rev. Lett.* **65** (1990) 2963
67. B.W. Lynn, M.E. Peskin, R.G. Stuart, in: *Physics with LEP*, eds. J. Ellis and R. Peccei, CERN 86-02 (1986)
68. B. Holdom, J. Terning, *Phys. Lett.* **B247** (1990) 88;
M. Golden, L. Randall, *Nucl. Phys.* **B361** (1991) 3;
C. Roiesnel, T.N. Truong, *Phys. Lett.* **B256** (1991) 439
69. D. Toussaint, *Phys. Rev.* **D18** (1978) 1626;
J.M Frere, J. Vermaseren, *Z. Phys.* **C19** (1983) 63
70. S. Bertolini, *Nucl. Phys.* **B272** (1986) 77;
W. Hollik, *Z. Phys.* **C32** (1986) 291; *Z. Phys.* **C37** (1988) 569
71. A. Denner, R. Guth, J.H. Kühn, *Phys. Lett.* **B240** (1990) 438
72. A. Denner, R. Guth, W. Hollik, J.H. Kühn, *Z. Phys.* **C51** (1991) 695
73. G. Passarino, Torino preprint DFTT/G-90-3 (1990);
G. Passarino, in *Proceedings of the XXVIth Rencontre de Moriond: '91 Electroweak Interactions and Unified Theories*, ed. Tran Thanh Van
74. B.W. Lynn, E. Nardi, *Nucl. Phys.* **B381** (1992) 467
75. G. Altarelli et al., *Nucl. Phys.* **B342** (1990) 15; *Phys. Lett.* **B245** (1990) 669;
M.C. Gonzalez-Garcia, J.W.F. Valle, *Phys. Lett.* **B259** (1991) 365;
J. Layssac, F.M. Renard, C. Verzegnassi, *Z. Phys.* **C53** (1992) 97
76. F. del Aguila, W. Hollik, J.M. Moreno, M. Quiros, *Nucl. Phys.* **B372** (1992) 3